# Accurate Finite Width Correction Functions for a Crack at a Straight Shank Hole

by

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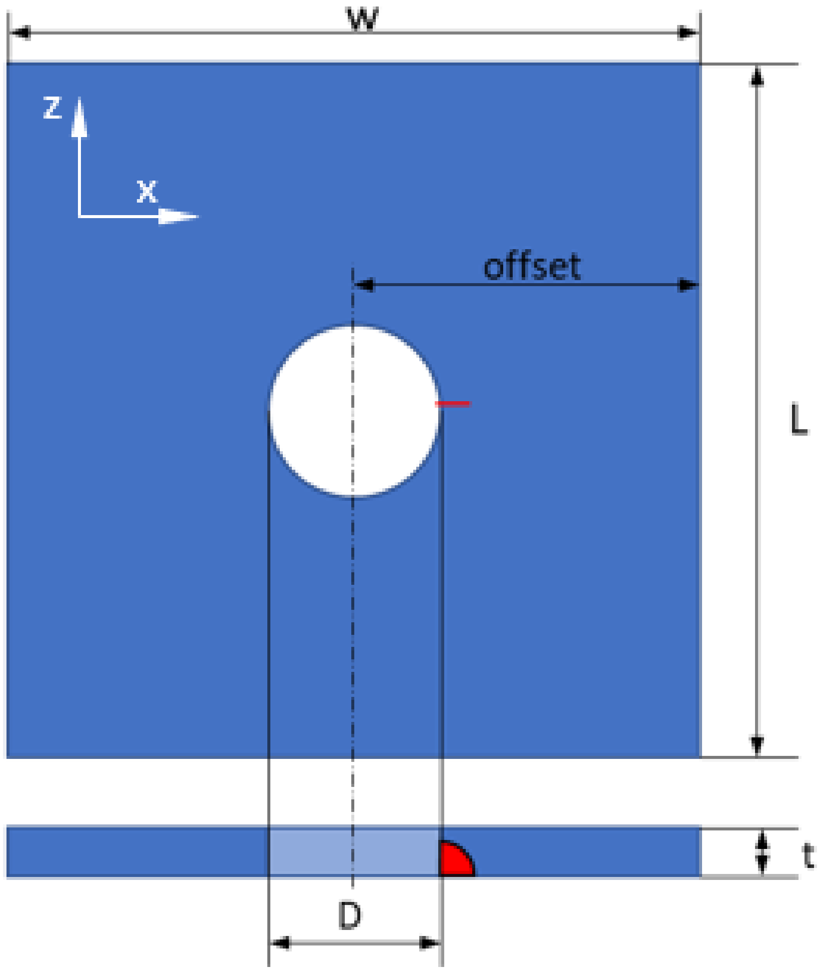
# Kick-off event for this pilot study, an ESRI Round Robin

## Engineered Residual Stress Implementation (ERSI)
## Stress Intensity Comparisons Round Robin ,June 2022




Geometry:

For finite-width plate L = 3W

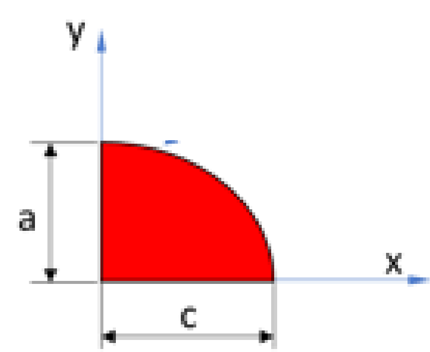
Material Properties:

E = 10.4e6

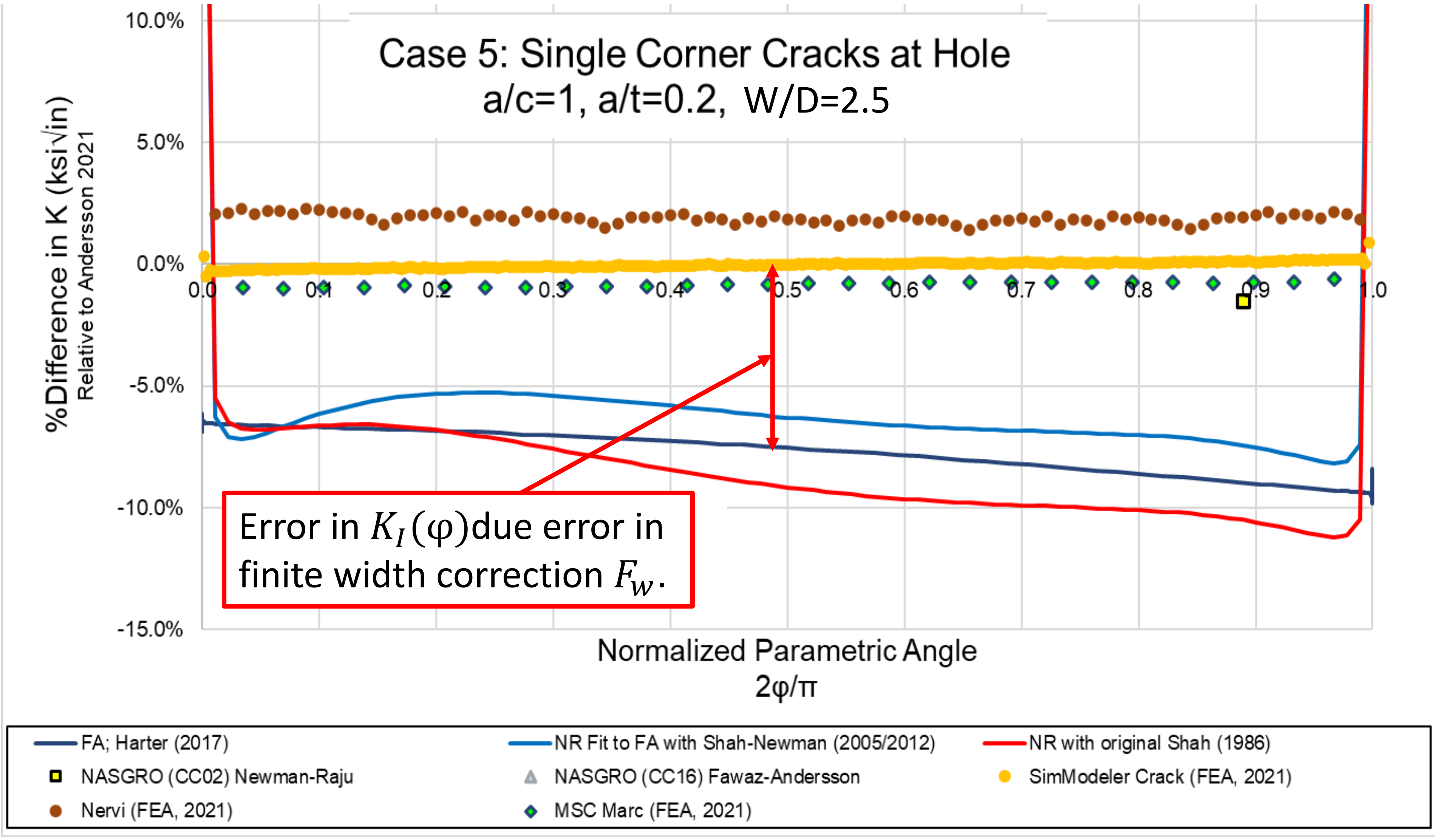
ν = 0.3

Loading:

Uniform Tension Stress = 10ksi



# Benchmark Report  R Pilarczyk, June 2022.



## The Classical Finite Width Correction $F_w$ Used.

Jim Newman's finite width correction function, for a single crack at a straight shank hole (from 1986) reads,

$$F_{w,Newman} = \left( \left( \frac{1}{\cos\left(\frac{\pi}{\frac{W}{R}}\right)} \right) \cdot \left( \frac{1}{\cos\left(\pi \cdot \frac{2 \cdot R + c}{2 \cdot W - 2 \cdot c} \cdot \sqrt{\frac{a}{t}}\right)} \right) \right)^{\frac{1}{2}}$$

This expression should ideally scale well $K_I\left(\frac{D}{t}, \frac{W}{D}, \frac{a}{t}, \frac{a}{c}\right)$ near both vertices 'a' and 'c' in the large 4D parameter space.

The above benchmark example shows that the error might be as large as 10% at vertex 'a' for a standard geometry, i.e. $D/t$=2, $a/c$=1, $a/t$=0.2 and $W/D$=2.5.

**The Present Pilot Study.**

- No ingenious engineering judgement.
- **Approach:** pure mining in numerical $K_I$-data, no mechanics whatsoever.
- **Start:** Derive 86000 $K_I(\phi)$-solutions covering a large $\left(\frac{D}{t},\frac{W}{D},\frac{a}{t},\frac{a}{c}\right)$-space. High accuracy in all computed data are of key importance in the present approach.
- **Work+Luck:** Identify accurate closed form $F_w$-expressions.
- **Verification:** The accuracy of all closed form $F_w$-equations are determined in the entire $\left(\frac{D}{t},\frac{W}{D},\frac{a}{t},\frac{a}{c}\right)$-space.

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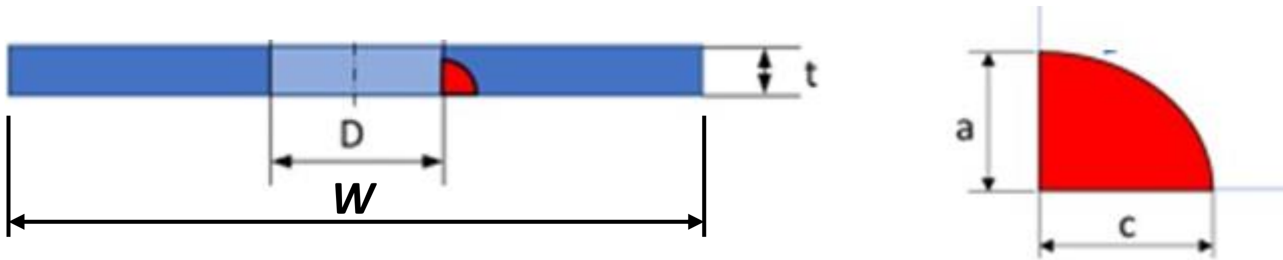
## The 86000 highly accurate $K_I(\phi)$-solutions covering a large $\left(\frac{D}{t}, \frac{W}{D}, \frac{a}{t}, \frac{a}{c}\right)$-space.




*D/2t*= **0.1**, 0.111, 0.125, 0.1428, 0.1667, 0.2, 0.25, 0.333, 0.4, 0.4444, 0.5, 0.5714, 0.667, 0.75, 0.8, 1.0, 1.25, 1.333, 1.5, 1.75, 2.0, 2.25, 2.5, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0, 9.0, **10.0**. (31 values)

*a/t*= **0.1**, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9 and **0.95**. (10 values)

*a/c*= **0.100**, 0.111, 0.125, 0.1428, 0.1667, 0.200, 0.250, 0.333, 0.500, 0.667, 0.750, 0.800, 1.000, 1.250, 1.333, 1.500, 2.000, 3.000, 4.000, 5.000, 6.000, 7.000, 8.000, 9.000, **10.000**. (25 values)

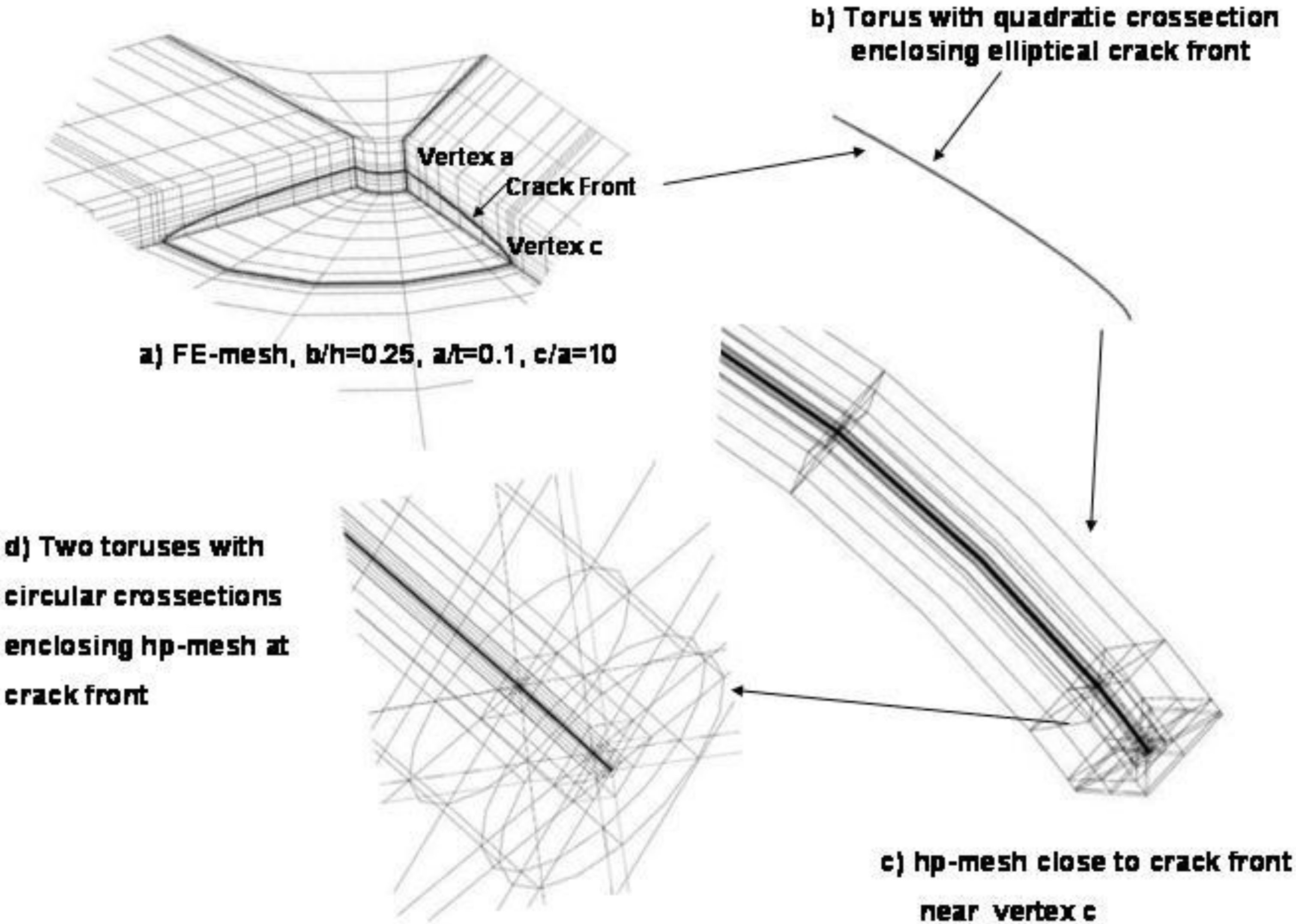
*W/D*= **1.6**, 1.8, 2.0, 2.2, 2.4, 2.8, 3.2, 3.6, 4.0, 4.6, 5.2, 5.8, 6.4, 7.0, 8.0,10.0,12.0,15.0,**100.0**. (19 values)

*H/W*=5.0

The constraint $\frac{D}{2} + 1.25 \cdot c \leq W/2$ resulted in about 86000 accurate $K_I(\phi)$-functions for tension, bending and pin loading.

**Employing a hp-version of FEM as implemented in the STRIPE-code.**

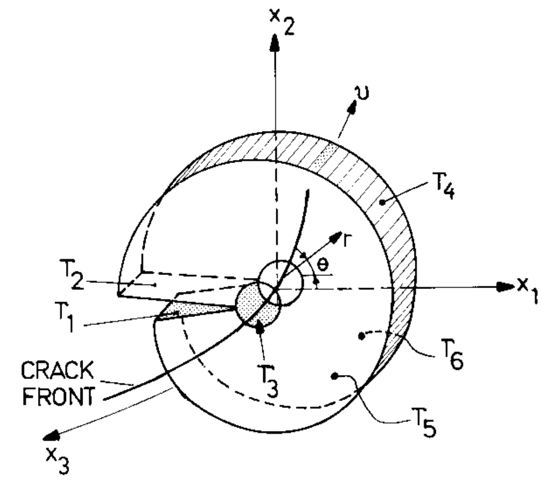
Mesh design:



# Accurate Calculation of $K_I(\phi)$ at arbitrary $\phi = \phi^*, 0 < \phi^* < \bar{\phi}$

The displacements **u** at a point $x_3$ on the crack front can be written

$$u(r,\theta,x_3) = \sum_{\alpha=I,II,III} K_\alpha(x_3) r^{1/2} \Psi_\alpha(\theta) + \text{smoother terms}$$




Domain $\Omega^e$ used for calculation of stress intensity factors

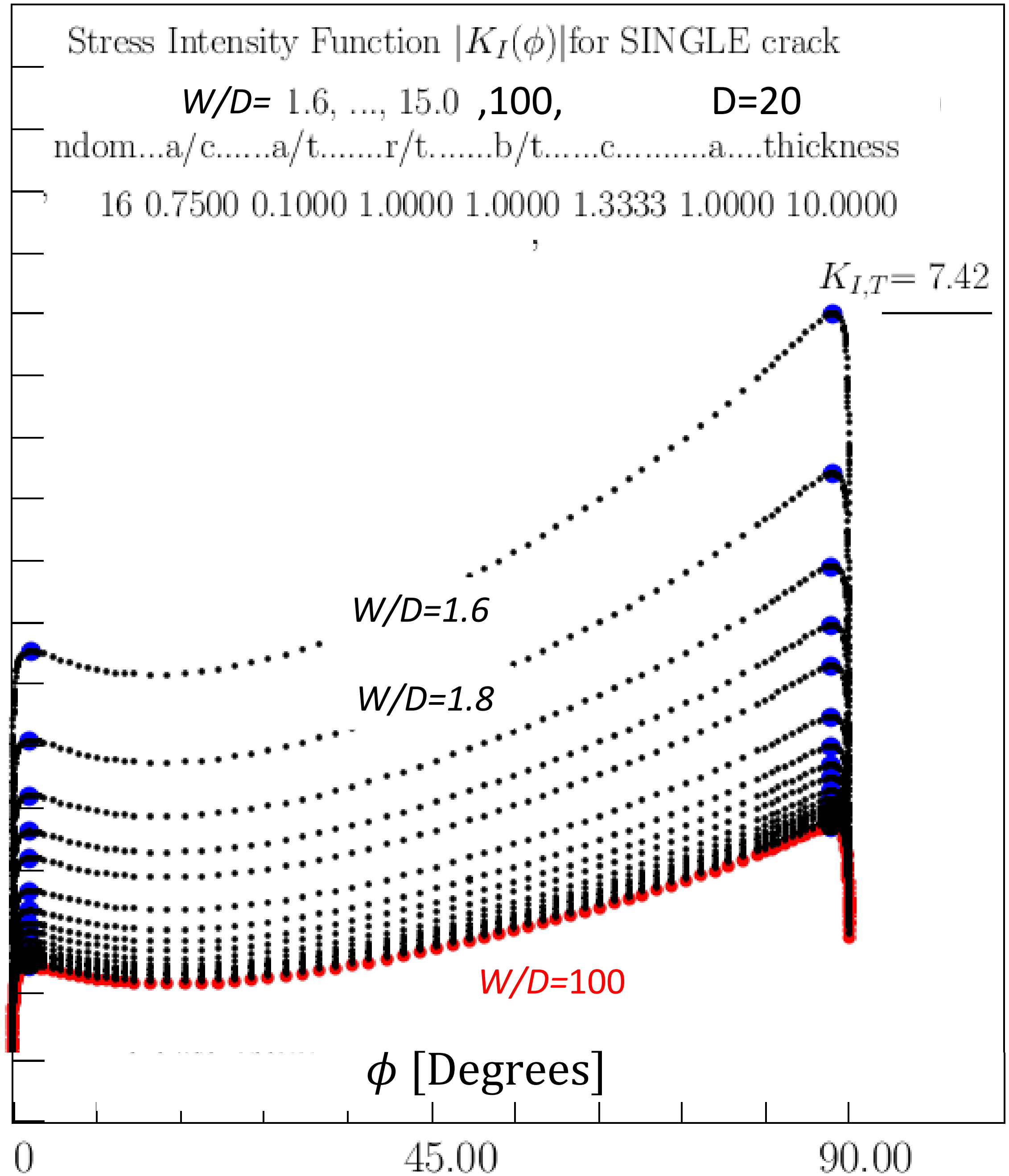
For smooth edges, the edge intensity functions $K_\alpha(x_3)$ are analytic on open intervals $s_k \le x_3 \le s_{k+1}$ . Hence, we approximate the edge intensity functions with the polynomials:

$$\bar{K}_\alpha(x_3) = \sum_{n=0}^{p} \bar{k}_{\alpha n} P_n(s), \qquad s = \frac{2(x_3 - s_k)}{s_{k+1} - s_k} - 1$$

Where $\bar{k}_{\alpha n}$ are unknown coefficients, $p$ is the polynomial order of the finite element trial functions, and $P_n$ the Legendre polynomials.

By applying the Maxwell-Betti reciprocity theorem the accuracy of the calculated $K$'s thus will depend only on a weighted average of the finite element solution inside the extraction domain. This gives *exponentially fast convergence*, with increasing $p$ to the exact $K$-values.

**19 out of 86000 $K_I$-solutions (for tension loading) where peak-values $K_{FEM-a}$ and $K_{FEM-c}$ are marked with blue disks.**



**Use of advanced mathematical vertex theory to find accurate peak-values of $K_I$ near vertices 'a' and 'c', that is, with relative errors in $K_I$ of order 0.03%.**

$K\_I(\phi)$ has near the two crack front vertices the following asymptotic expansion,

$$K(s) = \sum_{I=0}^{I=\infty} \sum_{i=1}^{i=\infty} S_{i,I} \cdot s^{\Lambda_i - 1/2 + I}$$

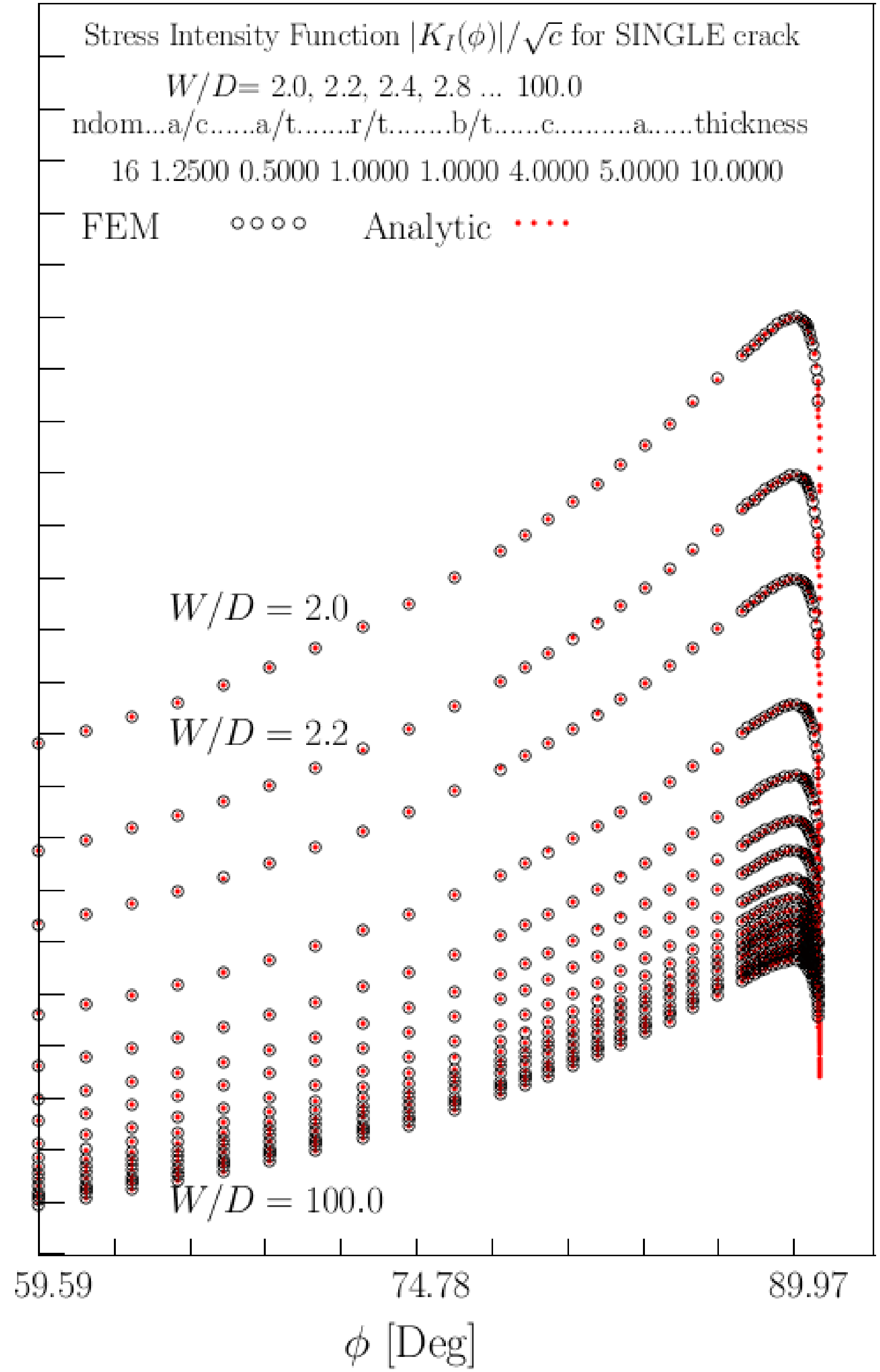
Where $S_{i,I}$ are unknown constants, $\Lambda_i$ known constants and $s(\phi)$ the crack front arc length measured from the actual vertex.
We here use only the four first terms, that is, in order of magnitude,

$$\Lambda_1 = 0.547841,$$
$$\Lambda_2 = 1.218267,$$
$$\Lambda_3 = \Lambda_1 + 1 = 1.547841,$$
$$\Lambda_4 = 1.681545.$$

The Figure exemplifies pointwise values of $K_I$ near vertex 'a' obtained from finite element analysis and the analytic expression, respectively.

Accurate peak values of $K_I$ are calculated by applying Newtons method to find the optimum of the analytic function $K(s)$. The relative error in $K_I$-peak at all 2x86000 vertices considered are estimated to be < 0.03%.

We now have access to 2x86000 accurate peak-values of the $K_I$ $\left(\frac{D}{t}, \frac{W}{D}, \frac{a}{t}, \frac{a}{c}\right)$-function. As W/D=100 corresponds to almost infinite width the corresponding $F_w$-values are easily obtained.



**Errors in J. Newman's 1986 $F_w$ formula at vertex 'a' for *R/t*=1 and *a/t*=0.2.**
(includes the above benchmark example *R/t*=1, *a/t*=0.2 and *W/D*=2.5).

The error is defined as,

$$Fw\_Err\_a\left(\frac{W}{D},\frac{r}{t},\frac{a}{t},\frac{a}{c}\right) = \left|\frac{K_{FEM}(\frac{W}{D})}{K_{FEM}(\frac{W}{t}=100)} - FW_{Newman}\right| / FW_{Newman}$$

a/c

| W/D | 0.1 | | | | | 0.2 | | | 0.5 | | | | 1.0 | | | | 2.0 | | | 5.0 | | | | | 10.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 100.0 | -0.0 | -0.0 | -0.0 | -0.0 | -0.0 | -0.0 | -0.0 | -0.0 | -0.0 | -0.0 | -0.0 | -0.0 | -0.0 | -0.0 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | -0.0 | -0.0 | -0.0 | -0.0 | -0.0 |
| 15.0 | 0.0 | 0.0 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.2 | 0.2 | 0.2 | 0.2 | 0.2 | 0.2 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | 0.1 | 0.1 | 0.1 | 0.1 | 0.2 |
| 12.0 | 0.0 | 0.1 | 0.1 | 0.1 | 0.2 | 0.2 | 0.2 | 0.2 | 0.2 | 0.3 | 0.3 | 0.3 | 0.3 | 0.3 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | 0.2 | 0.2 | 0.2 | 0.2 | 0.3 |
| 10.0 | 0.0 | 0.1 | 0.2 | 0.2 | 0.2 | 0.3 | 0.3 | 0.3 | 0.4 | 0.4 | 0.4 | 0.4 | 0.4 | 0.4 | 0.4 | 0.4 | 0.4 | 0.4 | 0.4 | 0.4 | 0.4 | 0.4 | 0.4 | 0.4 | 0.4 |
| 8.0 | 0.0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.5 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.6 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.6 | 0.6 | 0.6 | 0.6 | 0.7 |
| 7.0 | 0.0 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.8 | 0.8 | 0.8 | 0.8 | 0.9 | 0.9 | 0.9 | 0.9 | 0.9 | 0.9 | 0.9 | 0.9 | 0.9 | 0.9 | 0.9 | 0.9 | 0.9 |
| 6.4 | -0.0 | 0.2 | 0.3 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.1 | 1.1 | 1.1 | 1.1 | 1.1 | 1.1 | 1.1 | 1.1 | 1.1 | 1.1 | 1.1 |
| 5.8 | -0.1 | 0.2 | 0.4 | 0.6 | 0.8 | 0.9 | 1.0 | 1.1 | 1.2 | 1.2 | 1.3 | 1.3 | 1.3 | 1.3 | 1.3 | 1.3 | 1.3 | 1.4 | 1.4 | 1.4 | 1.4 | 1.3 | 1.3 | 1.4 | 1.4 |
| 5.2 | -0.2 | 0.1 | 0.4 | 0.7 | 0.9 | 1.1 | 1.3 | 1.4 | 1.5 | 1.6 | 1.6 | 1.6 | 1.6 | 1.7 | 1.7 | 1.7 | 1.7 | 1.7 | 1.8 | 1.8 | 1.7 | 1.7 | 1.7 | 1.7 | 1.8 |
| 4.6 | -0.4 | 0.1 | 0.5 | 0.9 | 1.2 | 1.5 | 1.7 | 1.9 | 2.0 | 2.1 | 2.1 | 2.1 | 2.2 | 2.2 | 2.2 | 2.2 | 2.2 | 2.3 | 2.3 | 2.3 | 2.3 | 2.3 | 2.3 | 2.3 | 2.3 |
| 4.0 | -1.0 | -0.2 | 0.5 | 1.1 | 1.6 | 2.0 | 2.3 | 2.6 | 2.8 | 2.9 | 2.9 | 2.9 | 3.0 | 3.0 | 3.0 | 3.0 | 3.1 | 3.1 | 3.2 | 3.2 | 3.2 | 3.2 | 3.2 | 3.2 | 3.2 |
| 3.6 | | -0.5 | 0.5 | 1.3 | 2.0 | 2.5 | 3.0 | 3.3 | 3.6 | 3.7 | 3.7 | 3.7 | 3.8 | 3.8 | 3.9 | 3.9 | 3.9 | 4.0 | 4.1 | 4.1 | 4.1 | 4.1 | 4.0 | 4.1 | 4.1 |
| 3.2 | | | | 1.5 | 2.5 | 3.2 | 3.9 | 4.3 | 4.7 | 4.9 | 4.9 | 4.9 | 5.0 | 5.1 | 5.1 | 5.1 | 5.2 | 5.3 | 5.4 | 5.4 | 5.4 | 5.4 | 5.4 | 5.4 | 5.4 |
| 2.8 | | | | | | 4.4 | 5.3 | 6.0 | 6.5 | 6.7 | 6.8 | 6.8 | 6.9 | 7.0 | 7.1 | 7.1 | 7.2 | 7.3 | 7.4 | 7.5 | 7.5 | 7.5 | 7.5 | 7.5 | 7.5 |
| 2.4 | | | | | | | | 8.8 | 9.6 | 9.9 | 10 | 10 | 10 | 10 | 10 | 10 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 | 11 |
| 2.2 | | | | | | | | 11 | 12 | 12 | 13 | 13 | 13 | 13 | 13 | 13 | 13 | 13 | 13 | 14 | 14 | 14 | 14 | 14 | 14 |
| 2.0 | | | | | | | | 14 | 16 | 16 | 16 | 16 | 17 | 17 | 17 | 17 | 17 | 18 | 18 | 18 | 18 | 18 | 18 | 18 | 18 |
| 1.8 | | | | | | | | | 21 | 22 | 22 | 22 | 22 | 23 | 23 | 23 | 23 | 23 | 24 | 24 | 24 | 24 | 24 | 24 | 24 |
| 1.6 | | | | | | | | | | 31 | 31 | 32 | 32 | 32 | 32 | 32 | 33 | 33 | 33 | 33 | 34 | 34 | 34 | 34 | 34 |

Error less than 2%

Benchmark 10%

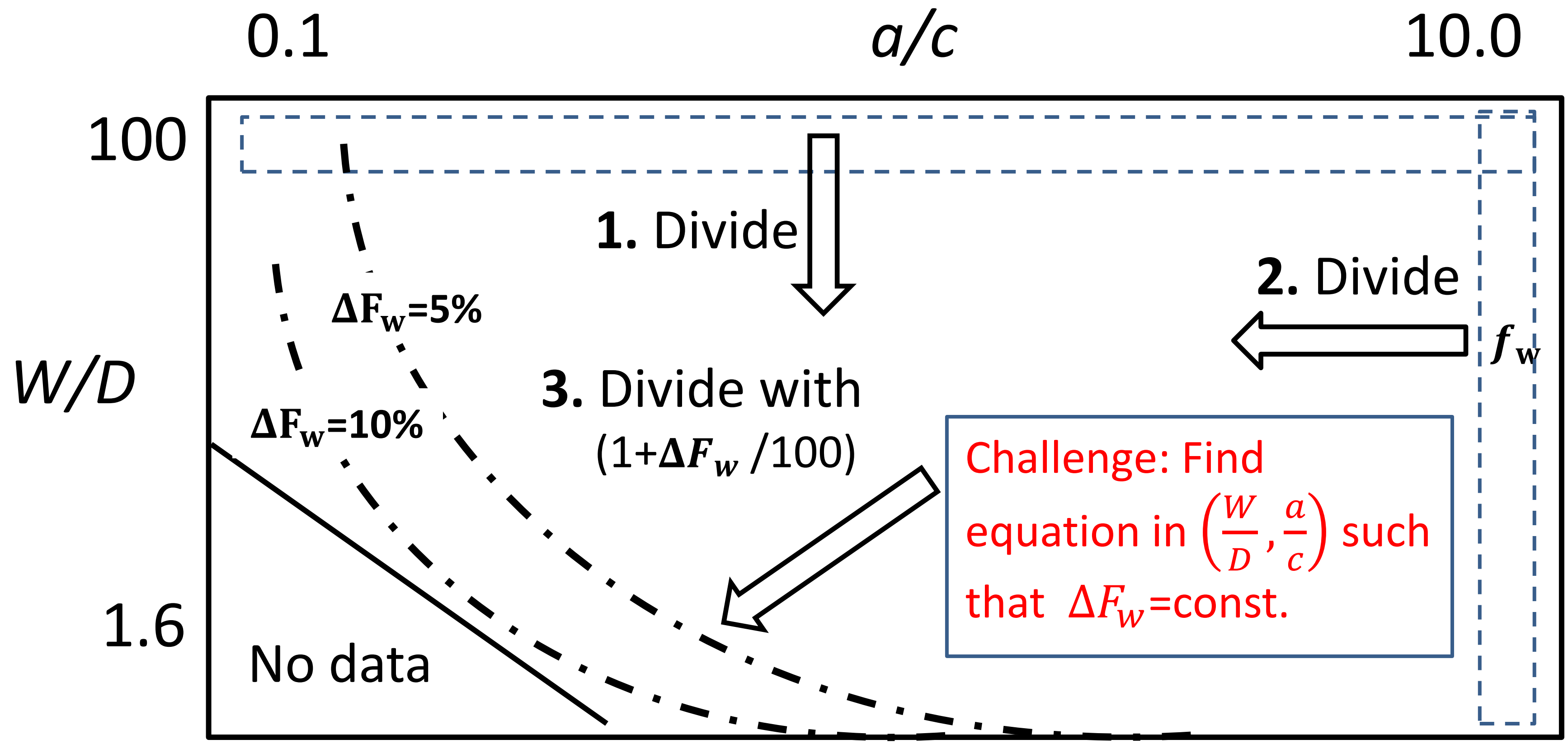
Error > 30 % for *W/D*=1.6.

Table Error in % in Jim Newman's $F_w$-formula (1986) for *R/t*=1.0 and *a/t*=0.2.

Results from the 86000 FEM analyses might be summarized in 620 tables of the type shown.

# Development of an almost exact Finite Width Correction Function $F_w\left(\frac{D}{t},\frac{W}{D},\frac{a}{t},\frac{a}{c}\right)$.

**Three Steps.**




One table, out of 620, with $K_I\left(\frac{W}{D},\frac{a}{c}\right)$ for *D/t* and *a/t* fixed.

# Development of an almost exact Finite Width Correction Function $F_w\left(\frac{D}{t},\frac{W}{D},\frac{a}{t},\frac{a}{c}\right)$.

*Assumption 1*: We know $K_I$ for infinite large plates (i.e. *W/D*=100 and plate height/*D*=250).
*Assumption 2*: Assume that $K_I$ also is known for *a/c*=10 (i.e. the data in the right column in tables of the type shown above). These data can be stored in very compact form as exemplified below.
*Assumption 3*: The remaining error is given by a simple analytic function $\Delta F_w$.

The table below exemplifies the error obtained when adopting assumptions 1 and 2 for the case *R/t*=1.0 and *a/t*=0.5.

*a/c* →

*W/D*

| W/D | 0.100 | 0.111 | 0.125 | 0.1428 | 0.1667 | 0.200 | 0.250 | 0.333 | 0.500 | 0.667 | 0.750 | 0.800 | 1.000 | 1.250 | 1.333 | 1.500 | 2.000 | 3.000 | 4.000 | 5.000 | 6.000 | 7.000 | 8.000 | 9.000 | 10.000 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 100.0 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 15.0 | 1.006 | 1.005 | 1.005 | 1.004 | 1.004 | 1.003 | 1.002 | 1.002 | 1.001 | 1.001 | 1.001 | 1.001 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 12.0 | 1.010 | 1.009 | 1.008 | [illegible] | 1.006 | 1.005 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | 1.001 | 1.001 | 1.001 | 1.000 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] |
| 10.0 | 1.016 | 1.014 | 1.012 | 1.011 | [illegible] | 1.007 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | 1.001 | 1.001 | 1.001 | 1.000 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] |
| 8.0 | 1.027 | 1.024 | 1.021 | 1.018 | 1.015 | 1.012 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | 1.001 | 1.001 | 1.001 | 1.001 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] |
| 7.0 | | | 1.029 | 1.024 | 1.020 | 1.016 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | 1.002 | 1.002 | 1.001 | 1.001 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] |
| 6.4 | | | | 1.030 | 1.025 | 1.020 | 1.015 | 1.011 | 1.007 | 1.005 | 1.004 | 1.004 | 1.003 | 1.002 | 1.002 | 1.001 | 1.001 | 1.001 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 5.8 | | | | | | 1.025 | 1.019 | 1.014 | 1.008 | 1.006 | 1.005 | 1.004 | 1.003 | 1.002 | 1.002 | 1.002 | 1.001 | 1.001 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 5.2 | | | | | | | 1.025 | 1.018 | 1.011 | 1.007 | 1.006 | 1.006 | 1.004 | 1.003 | 1.003 | 1.002 | 1.001 | 1.001 | 1.001 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 4.6 | | | | | | | 1.035 | 1.024 | 1.014 | 1.010 | [illegible] | 1.008 | 1.006 | 1.004 | 1.004 | 1.003 | 1.002 | 1.001 | 1.001 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 4.0 | | | | | | | 1.050 | 1.034 | 1.020 | 1.014 | 1.012 | 1.011 | 1.008 | 1.006 | 1.005 | 1.004 | 1.002 | 1.001 | 1.001 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 3.6 | | | | | | | 1.069 | 1.046 | 1.027 | 1.018 | 1.015 | 1.014 | 1.010 | 1.007 | 1.007 | 1.005 | 1.003 | 1.001 | 1.001 | 1.001 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 3.2 | | | | | | | | 1.065 | 1.037 | 1.025 | 1.021 | 1.019 | 1.014 | 1.010 | 1.009 | 1.007 | 1.004 | 1.002 | 1.001 | 1.001 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 2.8 | | | | | | | | | 1.056 | 1.037 | 1.031 | 1.029 | 1.021 | 1.015 | 1.012 | 1.011 | 1.006 | 1.003 | 1.002 | 1.001 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 2.4 | | | | | | | | | | 1.061 | 1.052 | 1.047 | 1.034 | 1.024 | 1.022 | 1.018 | 1.011 | 1.005 | 1.003 | 1.002 | 1.001 | 1.000 | 1.000 | 1.000 | 1.000 |
| 2.2 | | | | | | | | | | | | 1.064 | 1.046 | 1.033 | 1.029 | 1.024 | 1.015 | 1.007 | 1.004 | 1.002 | 1.001 | 1.001 | 1.000 | 1.000 | 1.000 |
| 2.0 | | | | | | | | | | | | | | 1.048 | 1.043 | 1.036 | 1.022 | 1.010 | 1.006 | 1.003 | 1.002 | 1.001 | 1.000 | 1.000 | 1.000 |
| 1.8 | | | | | | | | | | | | | | | | | 1.036 | 1.018 | 1.010 | 1.006 | 1.004 | 1.002 | 1.001 | 1.000 | 1.000 |
| 1.6 | | | | | | | | | | | | | | | | | 1.070 | 1.035 | 1.020 | 1.012 | 1.007 | 1.004 | 1.003 | 1.001 | 1.000 |

Error < 1% in this area

Error < 0.1% in this corner

Data in this corner of the table corresponds D/2+$c > W/2$.

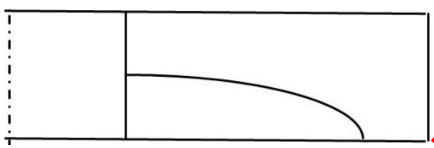
Next slide

# Errors $\Delta F_w$ in % for case $\frac{R}{t} = 1.0, \frac{a}{t} = 0.5$ *a/c*

| W/D | 1/10 | 1/9 | 1/8 | 1/7 | 1/6 | 1/5 | 1/4 | 1/3 | 1/2 | 2/3 | 3/4 | 4/5 | 1 | 5/4 | 4/3 | 3/2 | 2 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 10.0 | 1.6 | 1.4 | 1.2 | 1.1 | 0.9 | 0.7 | 0.6 | 0.4 | 0.3 | 0.2 | 0.2 | 0.2 | 0.1 | 0.1 | 0.1 | 0.1 | 0.0 |
| 8.0 | 2.7 | 2.4 | 2.1 | 1.8 | 1.5 | 1.2 | 0.9 | 0.7 | 0.4 | 0.3 | 0.2 | 0.2 | 0.2 | 0.1 | 0.1 | 0.1 | 0.1 |
| 7.0 | | | 2.9 | 2.4 | 2.0 | 1.6 | 1.3 | 0.9 | 0.6 | 0.4 | 0.3 | 0.3 | 0.2 | 0.2 | 0.2 | 0.1 | 0.1 |
| 6.4 | | | | 3.0 | 2.5 | 2.0 | 1.5 | 1.1 | 0.7 | 0.5 | 0.4 | 0.4 | 0.3 | 0.2 | 0.2 | 0.1 | 0.1 |
| 5.8 | | | | | | 2.5 | 1.9 | 1.4 | 0.8 | 0.6 | 0.5 | 0.4 | | | | 0.2 | 0.1 |
| 5.2 | | | | | | | 2.5 | 1.8 | 1.1 | 0.7 | 0.6 | 0.6 | 0.4 | 0.3 | 0.3 | 0.2 | 0.1 |
| 4.6 | | | | | | | 3.5 | 2.4 | 1.4 | 1.0 | 0.8 | 0.8 | 0.6 | 0.4 | 0.4 | 0.3 | 0.2 |
| 4.0 | | | | | | | 5.0 | 3.4 | 2.0 | 1.4 | 1.2 | 1.1 | 0.8 | 0.6 | 0.5 | 0.4 | 0.2 |
| 3.6 | | | | | | | 6.9 | 4.6 | 2.7 | 1.8 | 1.5 | 1.4 | 1.0 | 0.7 | 0.7 | 0.5 | 0.3 |
| 3.2 | | | | | | | | 6.5 | 3.7 | 2.5 | 2.1 | 1.9 | 1.4 | 1.0 | 0.9 | 0.7 | 0.4 |
| 2.8 | | | | | | | | | 5.6 | 3.7 | 3.1 | 2.9 | 2.1 | 1.5 | 1.3 | 1.1 | 0.6 |
| 2.4 | | | | | | | | | | 6.1 | 5.2 | 4.7 | 3.4 | 2.4 | 2.2 | 1.8 | 1.1 |
| 2.2 | | | | | | | | | | | | 6.4 | 4.6 | 3.3 | 2.9 | 2.4 | 1.5 |
| 2.0 | | | | | | | | | | | | | | 4.8 | 4.3 | 3.6 | 2.2 |
| 1.8 | | | | | | | | | | | | | | | | | 3.6 |
| 1.6 | | | | | | | | | | | | | | | | | **7.0** |
| W/D | 1/10 | 1/9 | 1/8 | 1/7 | 1/6 | 1/5 | 1/4 | 1/3 | 1/2 | 2/3 | 3/4 | 4/5 | 1 | 5/4 | 4/3 | 3/2 | 2 |

Error < 0.1% in this corner

Error < 1%



Errors are up to 6-7% !

The function, $\Delta F_w\left(\frac{W}{D}, \frac{a}{c}\right)$ to be developed

Shall ideally remove this error!

**The function $\Delta F_w(\frac{W}{D}, \frac{a}{c})$ for *R/t*=1.0, *a/t*=0.5 in %.**

# Calculate Contour Levels for $\Delta F_w\left(\frac{W}{D},\frac{a}{c}\right) = \mathbf{constant}$

| W/D | 1/10 | 1/9 | 1/8 | 1/7 | 1/6 | 1/5 | 1/4 | 1/3 | 1/2 | 2/3 | 3/4 | 4/5 | 1 | 5/4 | 4/3 | 3/2 | 2 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 10.0 | 1.6 | 1.4 | 1.2 | 1.1 | 0.9 | 0.7 | 0.6 | 0.4 | 0.3 | 0.2 | 0.2 | 0.2 | 0.1 | 0.1 | 0.1 | 0.1 | 0.0 |
| 8.0 | 2.7 | 2.4 | 2.1 | 1.8 | 1.5 | 1.2 | 0.9 | 0.7 | 0.4 | 0.3 | 0.2 | 0.2 | 0.2 | 0.1 | 0.1 | 0.1 | 0.1 |
| 7.0 | | | 2.9 | 2.4 | 2.0 | 1.6 | 1.3 | 0.9 | 0.6 | 0.4 | 0.3 | 0.3 | 0.2 | 0.2 | 0.2 | 0.1 | 0.1 |
| 6.4 | | | | 3.0 | 2.5 | 2.0 | 1.5 | 1.1 | 0.7 | 0.5 | 0.4 | 0.4 | 0.3 | 0.2 | 0.2 | 0.1 | 0.1 |
| 5.8 | | | | | | 2.5 | 1.9 | 1.4 | 0.8 | 0.6 | 0.5 | 0.4 | | | | | |
| 5.2 | | | | | | | 2.5 | 1.8 | 1.1 | 0.7 | 0.6 | 0.6 | | | | | |
| 4.6 | | | | | | | 3.5 | 2.4 | 1.4 | 1.0 | 0.8 | 0.8 | 0.6 | 0.4 | 0.4 | 0.3 | 0.2 |
| 4.0 | | | | | | | 5.0 | 3.4 | 2.0 | 1.4 | 1.2 | 1.1 | 0.8 | 0.6 | 0.5 | 0.4 | 0.2 |
| 3.6 | | | | | | | 6.9 | 4.6 | 2.7 | 1.8 | 1.5 | 1.4 | 1.0 | 0.7 | 0.7 | 0.5 | 0.3 |
| 3.2 | | | | | | | | 6.5 | 3.7 | 2.5 | 2.1 | 1.9 | 1.4 | 1.0 | 0.9 | 0.7 | 0.4 |
| 2.8 | | | | | | | | | 5.6 | 3.7 | 3.1 | 2.9 | 2.1 | 1.5 | 1.3 | 1.1 | 0.6 |
| 2.4 | | | | | | | | | | 6.1 | 5.2 | 4.7 | 3.4 | 2.4 | 2.2 | 1.8 | 1.1 |
| 2.2 | | | | | | | | | | | | 6.4 | 4.6 | 3.3 | 2.9 | 2.4 | 1.5 |
| 2.0 | | | | | | | | | | | | | | 4.8 | 4.3 | 3.6 | 2.2 |
| 1.8 | | | | | | | | | | | | | | | | | 3.6 |
| 1.6 | | | | | | | | | | | | | | | | | 7.0 |
| W/D | 1/10 | 1/9 | 1/8 | 1/7 | 1/6 | 1/5 | 1/4 | 1/3 | 1/2 | 2/3 | 3/4 | 4/5 | 1 | 5/4 | 4/3 | 3/2 | 2 |

$\Delta F_w\left(\frac{W}{D},\frac{a}{c}\right) = 0.3\%.$

Calculate accurate locations of contour levels for $\Delta F_w\left(\frac{W}{D},\frac{a}{c}\right) = 0.1\%, 0.3\%, 0.5\%...$ errors from the accurate data in the table.

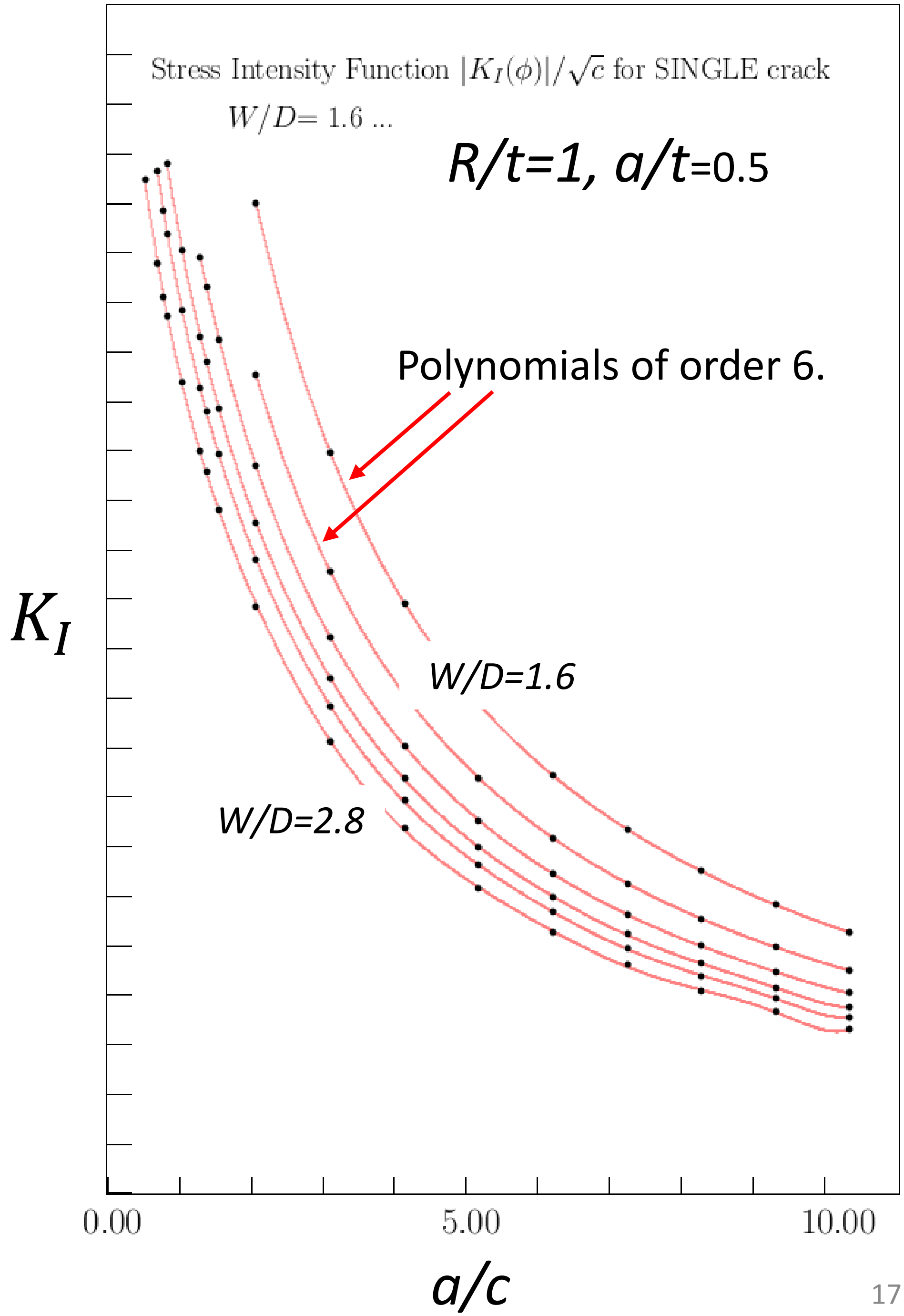
**The function $\Delta F_w(\frac{W}{D},\frac{a}{c})$ for *R/t*=1.0, *a/t*=0.5 in %.**

**Calculation of Contour Levels for** $\boldsymbol{K_I\left(\frac{W}{D},\frac{a}{c}\right) = \text{constant}}$.

All functions considered here, i.e. $K_I\left(\frac{D}{t},\frac{W}{D},\frac{a}{t},\frac{a}{c}\right), \Delta F_w\left(\frac{D}{t},\frac{W}{D},\frac{a}{t},\frac{a}{c}\right)$ are smooth analytic functions which can be accurately approximated by, for example, polynomials.

Finding contour levels for $K_I, \Delta F_w$ etc. is then very straight forward numerically.

The figure to right exemplifies how well polynomials of order 6 in *a/c* fits table data (the black dots) for the case *R/t*=1, *a/t*=0.5.



# Smoothnesss of $K_I\left(\frac{W}{D}, \frac{a}{t}, \frac{a}{c}\right)$

The figure to right exemplify how well polynomials of order 6 in *a/t* fits table data (the black dots) for the case *R/t*=1, *a/c*=2.0.

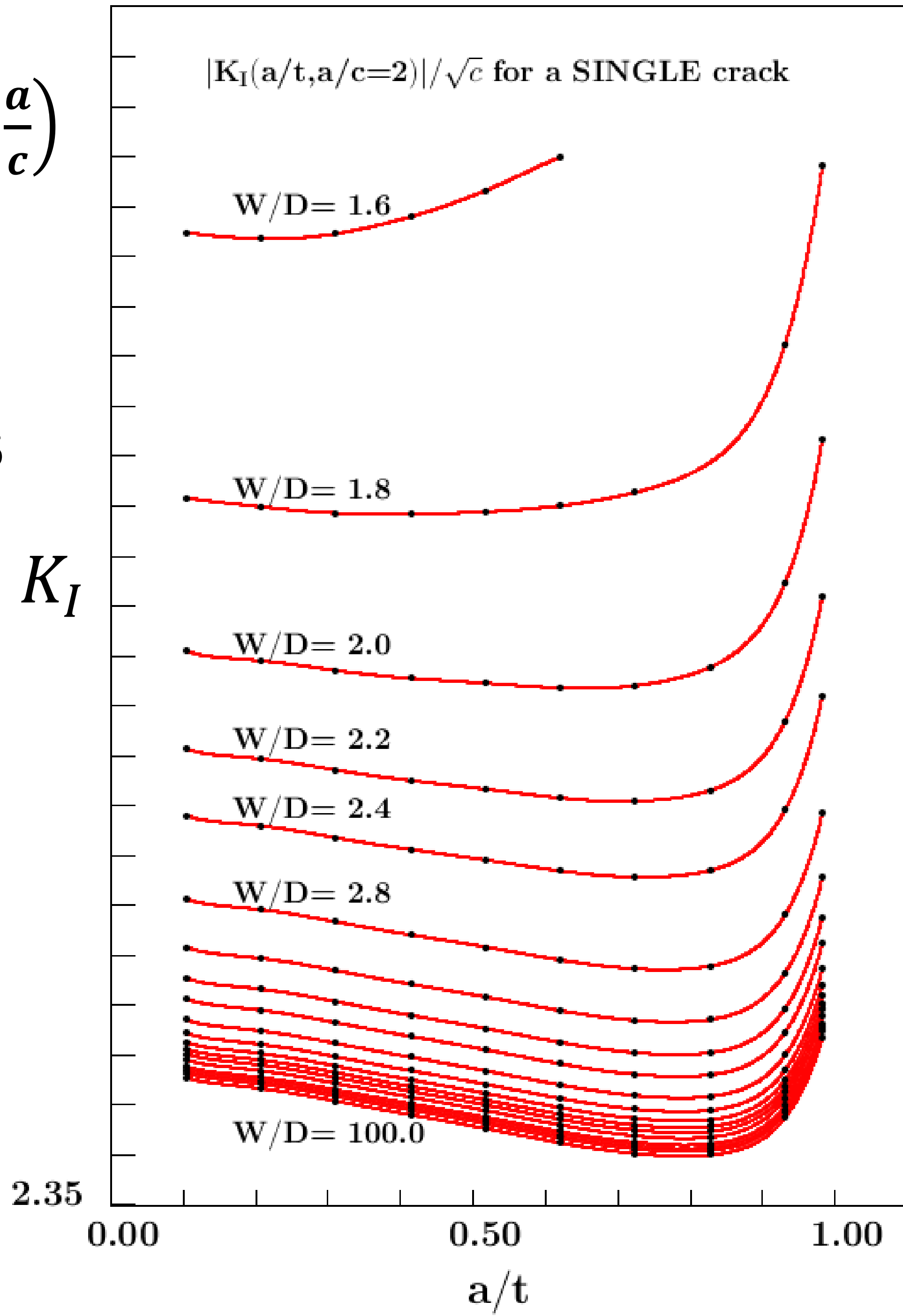
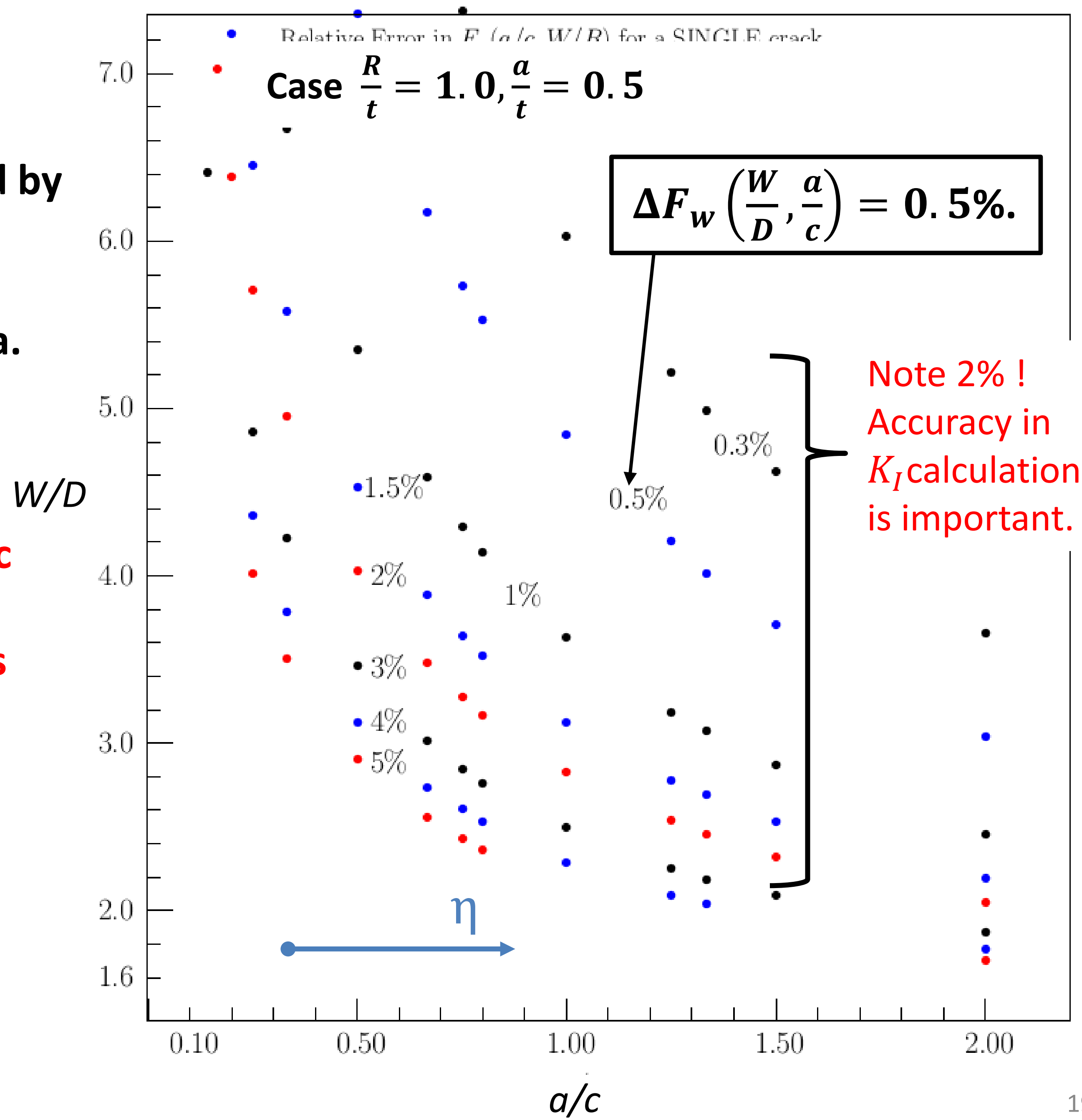


Figure shows level contours calculated by polynomial interpolation in available table data.




**Challenge:**
**Develop an analytic function that fit all red-blue-black dots with high accuracy.**

**Does points with $\Delta F_W = \text{const.}$ fit a function of type $\frac{1}{\eta}$ well?**

**A simple and very accurate equation in the variable $\Delta F_w$.**

$$\frac{a}{c} = c_1(\Delta F_w) \cdot \eta + c_2(\Delta F_w) \qquad (1)$$

$$\frac{W}{D} = \frac{c_3(\Delta F_w)}{\eta} + c_4(\Delta F_w) \qquad (2)$$

Eliminating $\eta$ from (1) and (2) gives the following dependence between $\frac{W}{D}$ and $\frac{a}{c}$,

$$\frac{W}{D} = c_1(\Delta F_w) \cdot \frac{c_3(\Delta F_w)}{\frac{a}{c} - c_2(\Delta F_w)} + c_4(\Delta F_w) (3)$$

Replace $c_1(\Delta F_w) \cdot c_3(\Delta F_w)$ with $c_5(\Delta F_w)$ and we get,

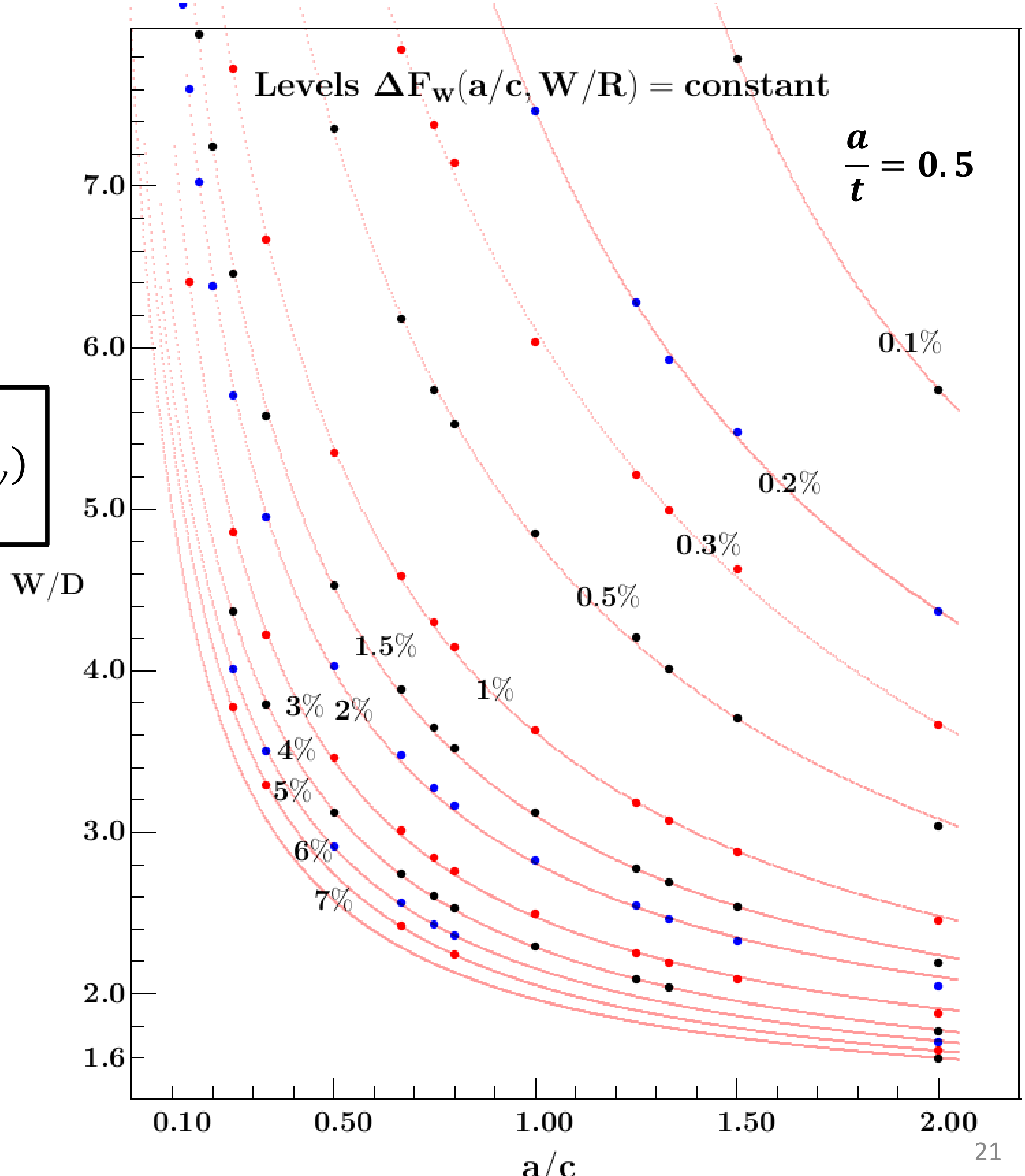
$$\frac{W}{D} = \frac{c_5(\Delta F_w)}{\frac{a}{c} - c_2(\Delta F_w)} + c_4(\Delta F_w) \qquad (4)$$

Equation (4) is, of reasons unknown to us, an equation that fits extremely well with all numerical data investigated so far (i.e. 12000 out of 86000 solutions).

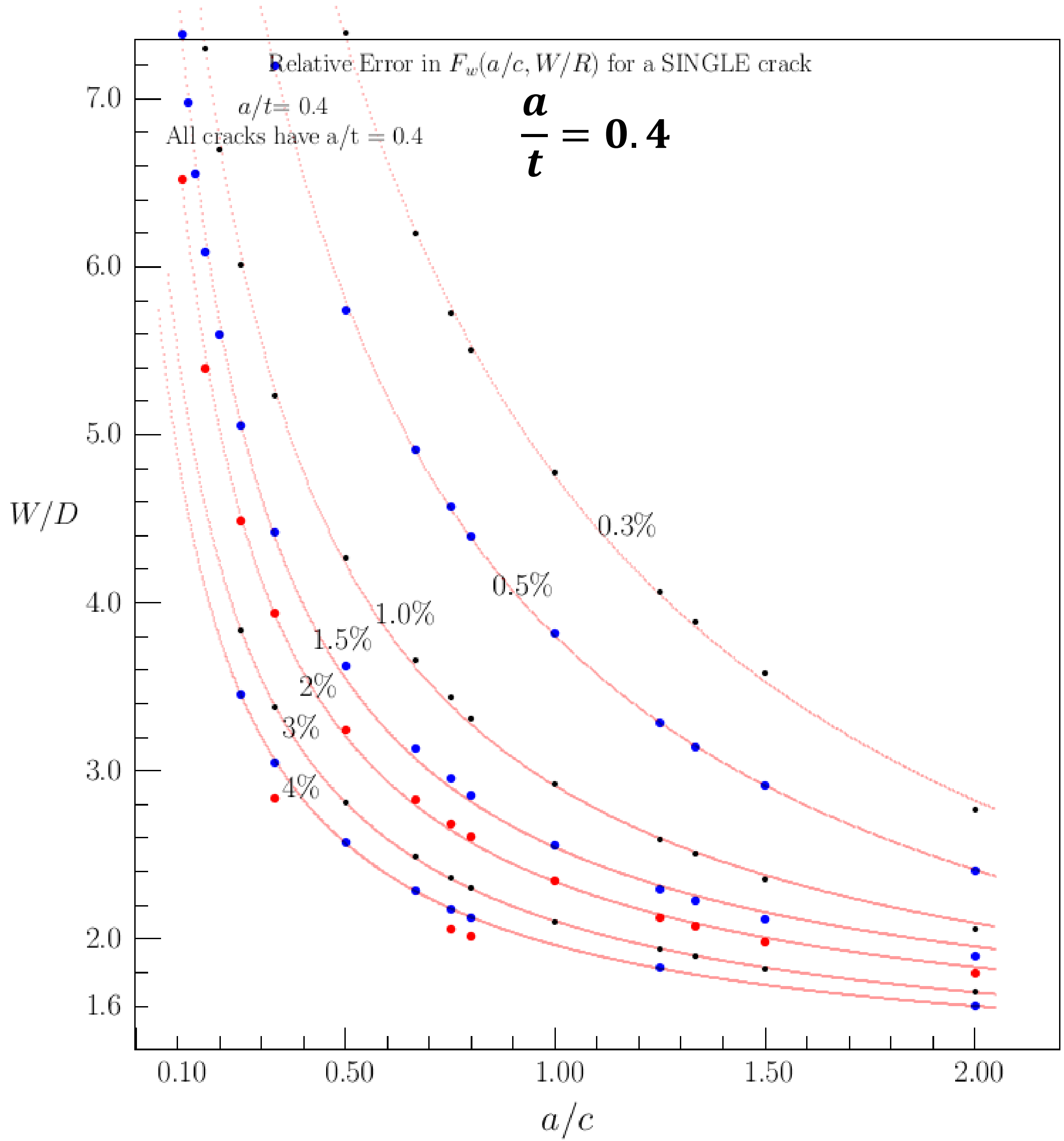
- The red-blue-black dots are errors obtained from the very accurate FEM solutions.
- The red curves are the equation,

$$\frac{W}{D} = \frac{c_5(\Delta F_w)}{\frac{a}{c} - c_2(\Delta F_w)} + c_4(\Delta F_w)$$

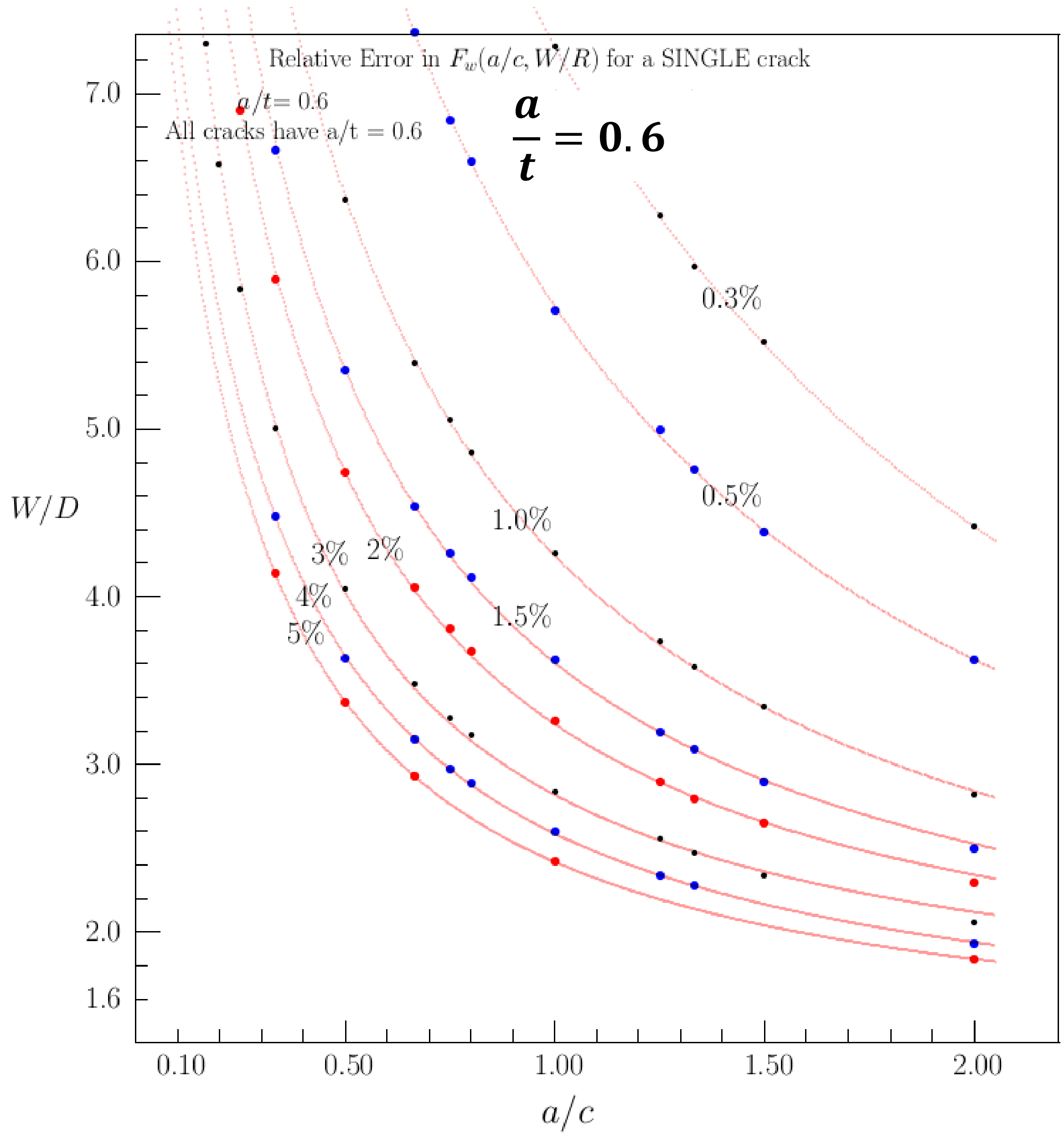
- The three coefficients $c_i(\Delta F_w)$ are determined by a least square fit to red-blue-black data points (weight-function=1.0).
- Agreement is excellent for $0.1\% \leq \Delta F_w \leq 7\%$.



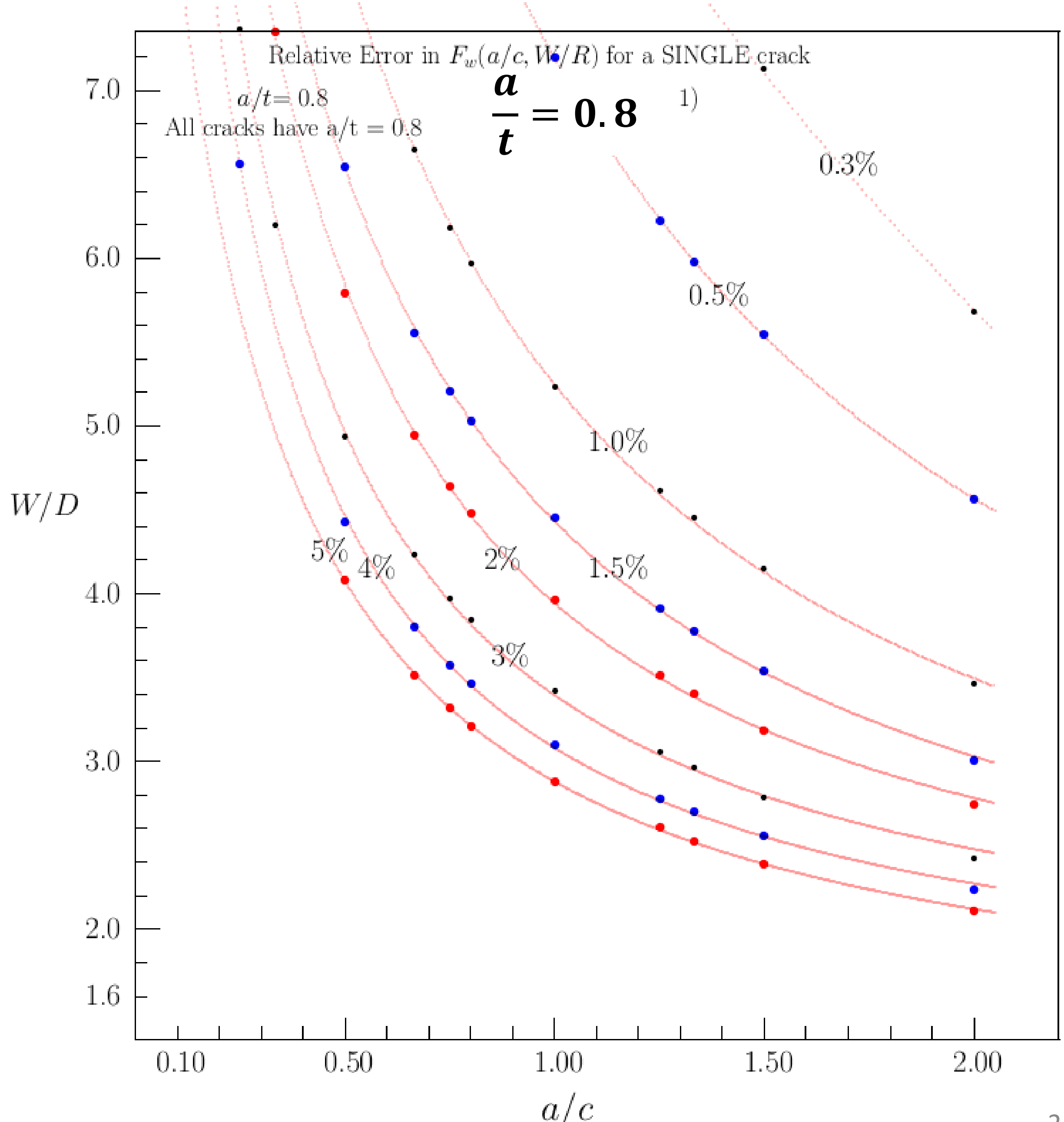
Graphs valid for **a/t**=0.4 showing $\Delta F_w = constant.$ Coefficients $\{c_i, i = 5,4,2\}$ are obtained from a least square fit to data (contour level curves) in the table of accurate $\Delta F_w$-values.



Graphs valid for **a/t**=0.6 showing $\Delta F_w = constant.$ Coefficients $\{c_i, i = 5,4,2\}$ are obtained from a least square fit to data (contour level curves) in the table of accurate $\Delta F_w$-values.



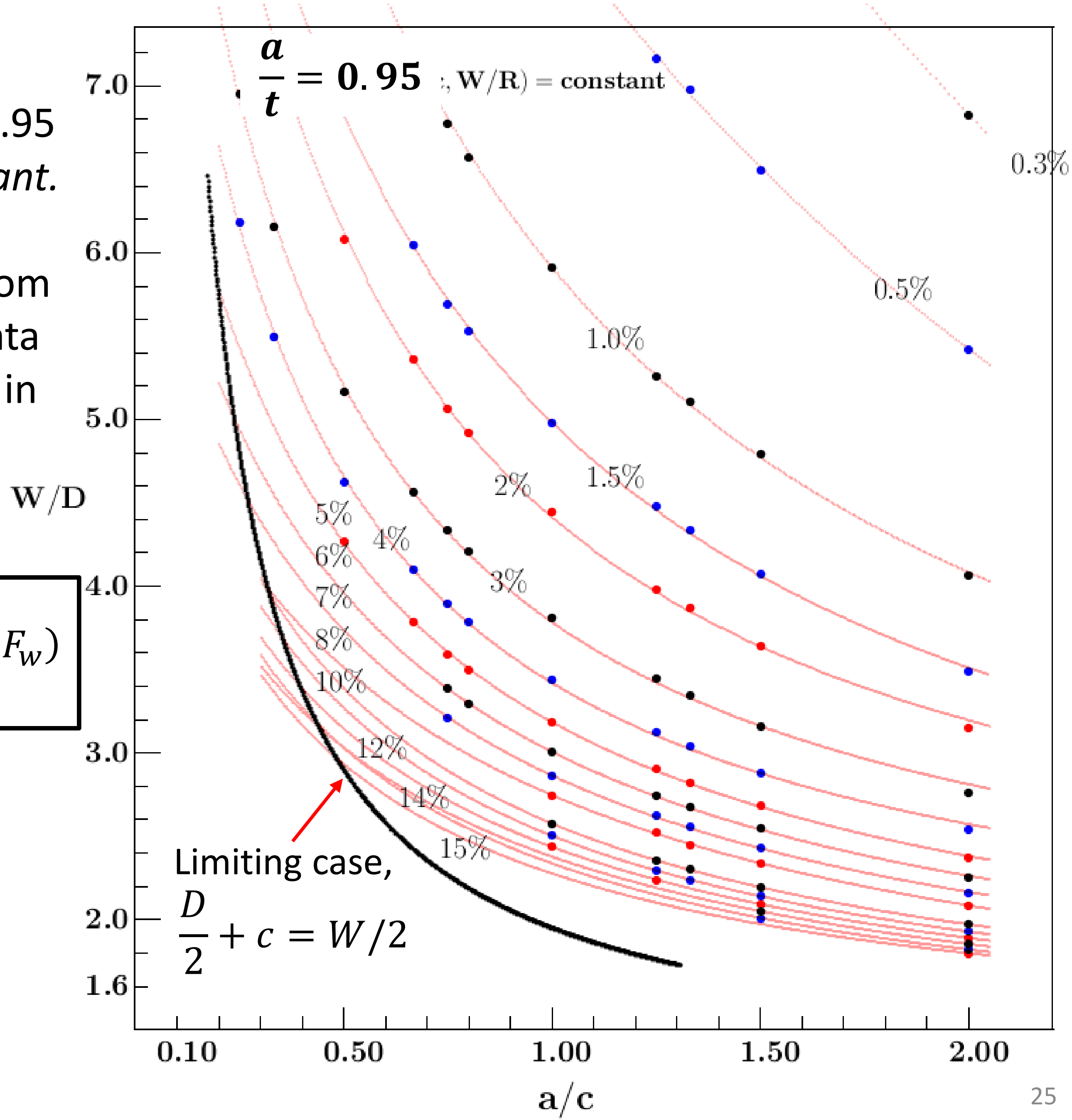
Graphs valid for **a/t**=0.8 showing $\Delta F_w = constant.$ Coefficients $\{c_i, i = 5,4,2\}$ are obtained from a least square fit to data (contour level curves) in the table of accurate $\Delta F_w$-values.



Graphs valid for **a/t**=0.95 showing $\Delta F_w = constant.$ Coefficients $\{c_i, i = 5,4,2\}$ are obtained from a least square fit to data (contour level curves) in the table of accurate $\Delta F_w$-values.

$$\frac{W}{D} = \frac{c_5(\Delta F_w)}{\frac{a}{c} - c_2(\Delta F_w)} + c_4(\Delta F_w)$$

Agreement is excellent for
$0.1\% \leq \Delta F_w \leq 15\%.$

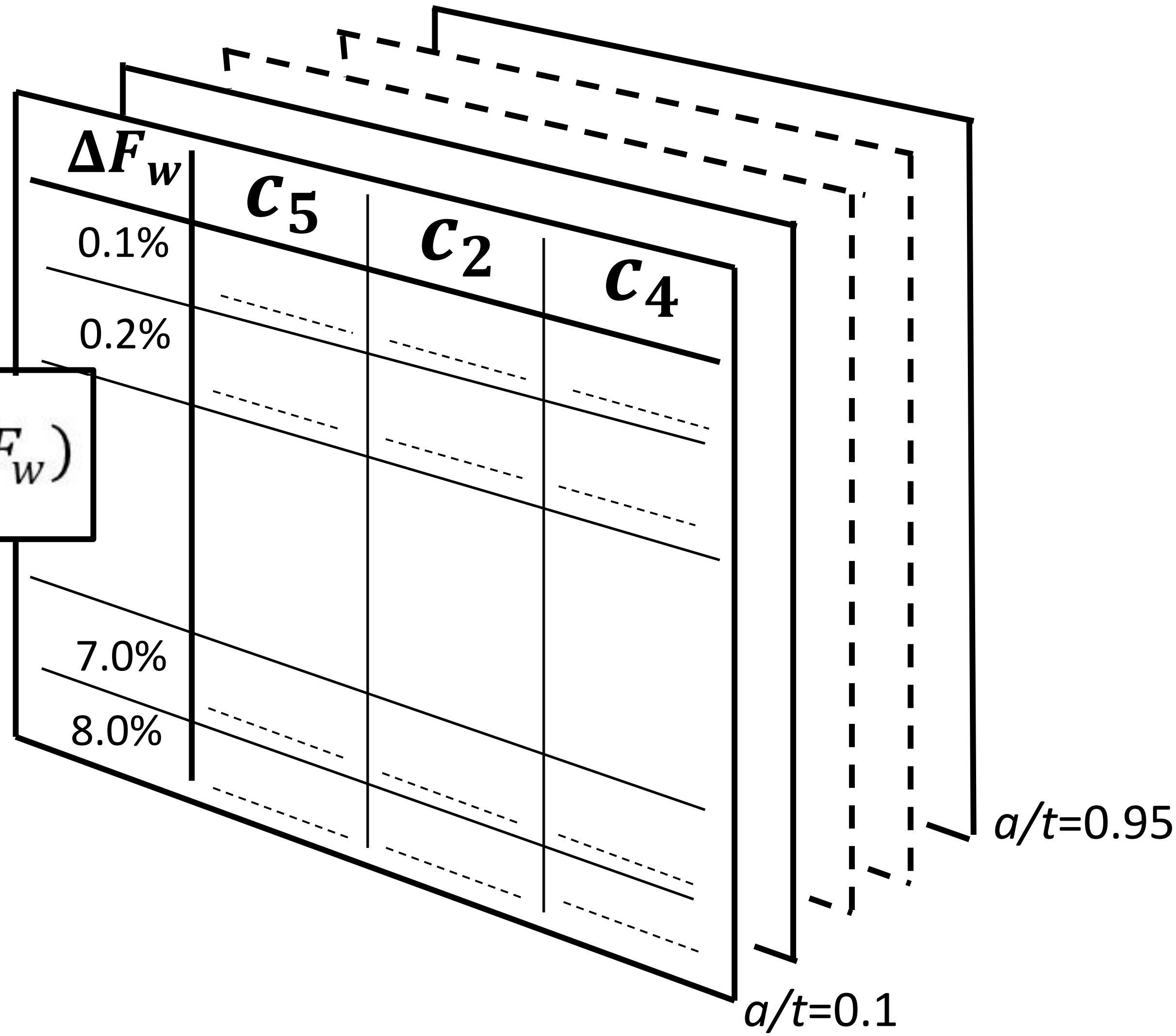


# Calculation of $\Delta F_w(\frac{W}{D}, \frac{a}{t}, \frac{a}{c})$ with high accuracy.

For fixed $(\frac{W}{D}, \frac{a}{t}, \frac{a}{c})$ calculate $\Delta F_w$ from,

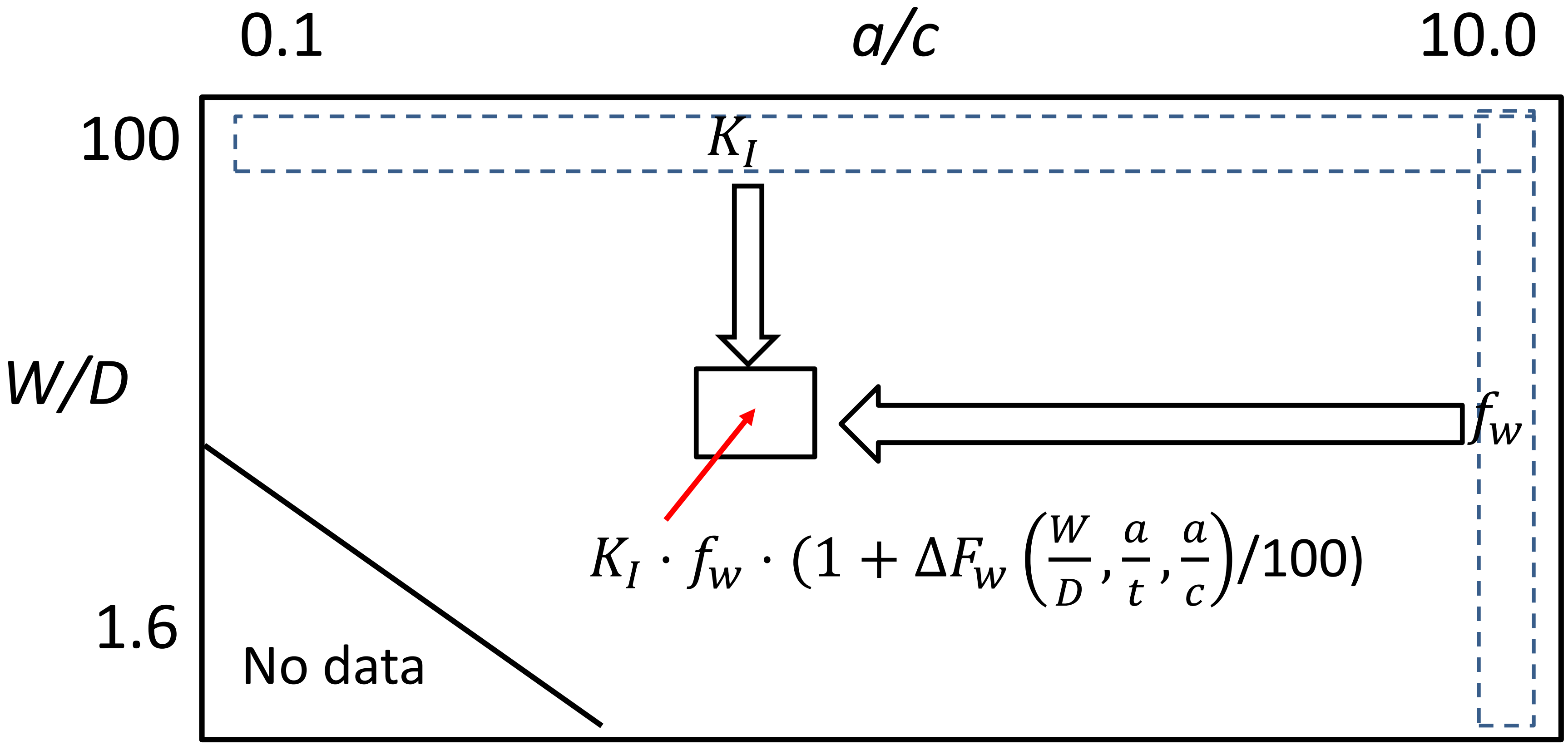
$$\frac{W}{D} = \frac{c_5(\Delta F_w)}{\frac{a}{c} - c_2(\Delta F_w)} + c_4(\Delta F_w)$$

and data in the 10 tables for *a/t*=0.1, 0.2,…, 0.95.



## The almost exact Function $F_w\left(\frac{D}{t},\frac{W}{D},\frac{a}{t},\frac{a}{c}\right)$.

We exemplify the high accuracy of $F_w$ by regenerating all tables with $K_I(\frac{W}{D},\frac{a}{c})$-data using the $F_w$-function. An entry in the table is obtained as sketched below.




**AFGROW/NASGRO** The scheme above can be used for compact storage of huge data bases for distribution to users .

# Accuracy of $\Delta F_w\left(\frac{W}{D},\frac{a}{c}\right)$ for the case $\frac{R}{t}=1.0, \frac{a}{t}=0.5$.

| W/D | 1/10 | 1/9 | 1/8 | 1/7 | 1/6 | 1/5 | 1/4 | 1/3 | 1/2 | 2/3 | 3/4 | 4/5 | 1 | 5/4 | 4/3 | 3/2 | 2 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 100.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 15.0 | 0.6 | 0.5 | 0.5 | 0.4 | 0.4 | 0.3 | 0.2 | 0.2 | 0.1 | 0.1 | 0.1 | 0.1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 12.0 | 1.0 | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 | 0.3 | 0.2 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.0 |
| 10.0 | 1.6 | 1.4 | 1.2 | 1.1 | 0.9 | 0.7 | 0.6 | 0.4 | 0.3 | 0.2 | 0.2 | 0.2 | 0.1 | 0.1 | 0.1 | 0.1 | 0.0 |
| 8.0 | 2.7 | 2.4 | 2.1 | 1.8 | 1.5 | 1.2 | 0.9 | 0.7 | 0.4 | 0.3 | 0.2 | 0.2 | 0.2 | 0.1 | 0.1 | 0.1 | 0.1 |
| 7.0 | | | 2.9 | 2.4 | 2.0 | 1.6 | 1.3 | 0.9 | 0.6 | 0.4 | 0.3 | 0.3 | 0.2 | 0.2 | 0.2 | 0.1 | 0.1 |
| 6.4 | | | | 3.0 | 2.5 | 2.0 | 1.5 | 1.1 | 0.7 | 0.5 | 0.4 | 0.4 | 0.3 | 0.2 | 0.2 | 0.1 | 0.1 |
| 5.8 | | | | | | 2.5 | 1.9 | 1.4 | 0.8 | 0.6 | 0.5 | 0.4 | 0.3 | 0.2 | 0.2 | 0.2 | 0.1 |
| 5.2 | | | | | | | 2.5 | 1.8 | 1.1 | 0.7 | 0.6 | 0.6 | 0.4 | 0.3 | 0.3 | 0.2 | 0.1 |
| 4.6 | | | | | | | 3.5 | 2.4 | 1.4 | 1.0 | 0.8 | 0.8 | 0.6 | 0.4 | 0.4 | 0.3 | 0.2 |
| 4.0 | | | | | | | 5.0 | 3.4 | 2.0 | 1.4 | 1.2 | 1.1 | 0.8 | 0.6 | 0.5 | 0.4 | 0.2 |
| 3.6 | | | | | | | 6.9 | 4.6 | 2.7 | 1.8 | 1.5 | 1.4 | 1.0 | 0.7 | 0.7 | 0.5 | 0.3 |
| 3.2 | | | | | | | | 6.5 | 3.7 | 2.5 | 2.1 | 1.9 | 1.4 | 1.0 | 0.9 | 0.7 | 0.4 |
| 2.8 | | | | | | | | | | 3.7 | 3.1 | 2.9 | 2.1 | 1.5 | 1.3 | 1.1 | 0.6 |
| 2.4 | | | | | | | | | | 6.1 | 5.2 | 4.7 | 3.4 | 2.4 | 2.2 | 1.8 | 1.1 |
| 2.2 | | | | | | | | | | | | 6.4 | 4.6 | 3.3 | 2.9 | 2.4 | 1.5 |
| 2.0 | | | | | | | | | | | | | | 4.8 | 4.3 | 3.6 | 2.2 |
| 1.8 | | | | | | | | | | | | | | | | | 3.6 |
| 1.6 | | | | | | | | | | | | | | | | | 7.0 |

Errors are up to 6-7% !

**Error in % before applying the $\Delta F_w\left(\frac{W}{D},\frac{a}{c}\right)$-correction.**

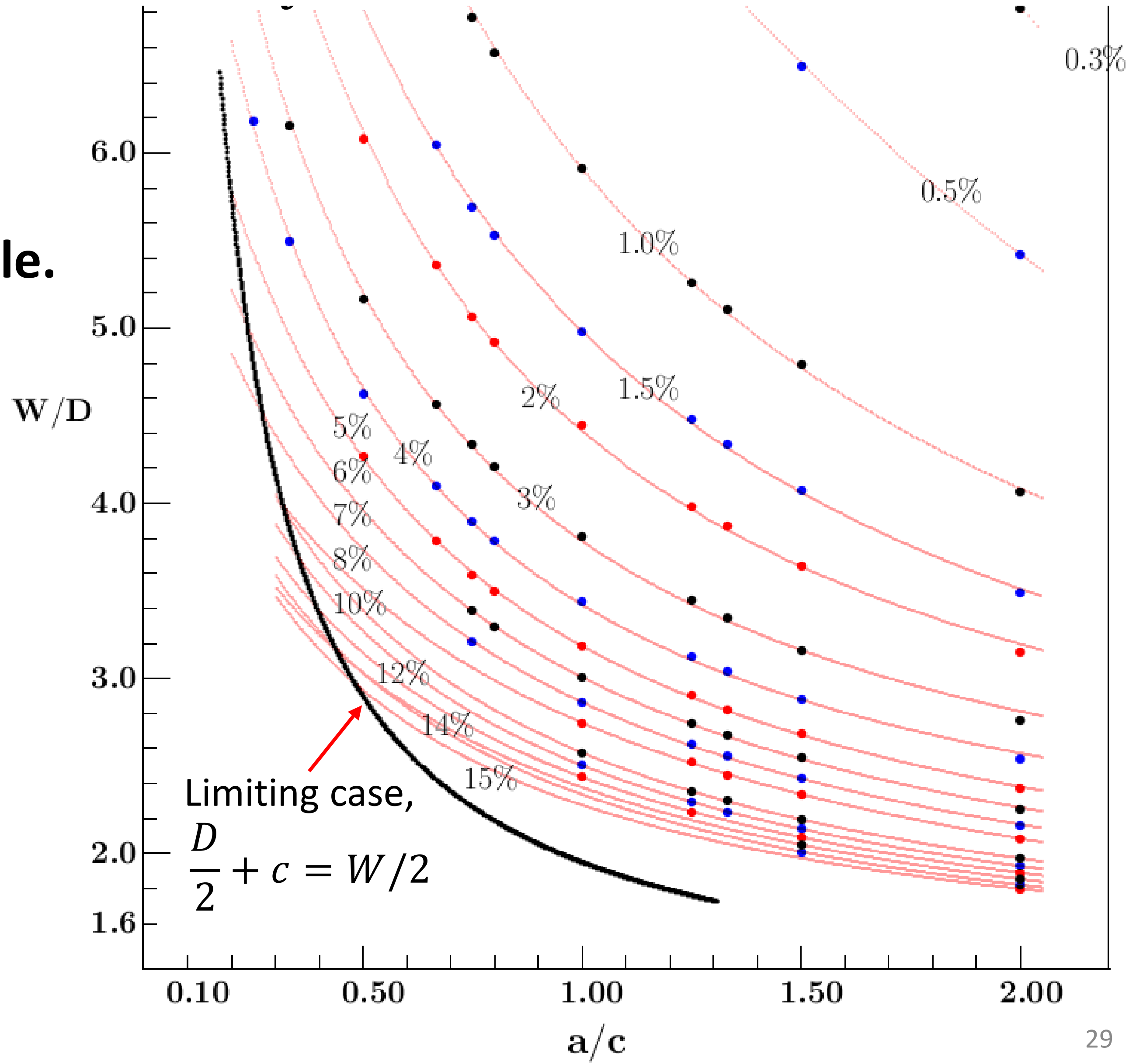
| W/D | 1/10 | 1/9 | 1/8 | 1/7 | 1/6 | 1/5 | 1/4 | 1/3 | 1/2 | 2/3 | 3/4 | 4/5 | 1 | 5/4 | 4/3 | 3/2 | 2 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 100.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 15.0 | 0.1 | 0.0 | 0.2 | 0.1 | 0.1 | 0.0 | 0.0 | 0.0 | 0.1 | 0.1 | 0.1 | 0.1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 12.0 | 0.0 | 0.2 | 0.1 | 0.1 | 0.1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.1 | 0.1 | 0.1 | 0.1 | 0.0 |
| 10.0 | 0.1 | 0.1 | 0.2 | 0.1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.1 | 0.1 | 0.1 | 0.0 |
| 8.0 | 0.2 | 0.1 | 0.1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.1 | 0.0 | 0.0 | 0.0 | 0.1 | 0.0 | 0.0 | 0.0 | 0.1 | 0.1 |
| 7.0 | | | 0.1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.1 |
| 6.4 | | | | 0.0 | 0.1 | 0.0 | 0.1 | 0.1 | 0.1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.1 |
| 5.8 | | | | | | 0.1 | 0.1 | 0.1 | 0.1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.1 | 0.1 | 0.0 | 0.1 |
| 5.2 | | | | | | | 0.1 | 0.1 | 0.0 | 0.1 | 0.1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.1 | 0.0 |
| 4.6 | | | | | | | 0.1 | 0.1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 4.0 | | | | | | | 0.0 | 0.1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 3.6 | | | | | | | 0.1 | 0.1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.1 | 0.1 | 0.0 | 0.0 |
| 3.2 | | | | | | | | 0.1 | 0.1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.1 | 0.0 |
| 2.8 | | | | | | | | | 0.2 | 0.0 | 0.1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.1 |
| 2.4 | | | | | | | | | | 0.1 | 0.1 | 0.0 | 0.0 | 0.1 | 0.1 | 0.1 | 0.1 |
| 2.2 | | | | | | | | | | | | 0.1 | 0.1 | 0.1 | 0.1 | 0.2 | 0.0 |
| 2.0 | | | | | | | | | | | | | | 0.1 | 0.1 | 0.1 | 0.2 |
| 1.8 | | | | | | | | | | | | | | | | | 0.2 |
| 1.6 | | | | | | | | | | | | | | | | | 0.0 |

Errors are 0.0 - 0.2% !

**Error in % after applying the $\Delta F_w\left(\frac{W}{D},\frac{a}{c}\right)$-correction.**

# Accuracy of $\Delta F_w\left(\frac{W}{D},\frac{a}{c}\right)$ for the case $\frac{R}{t}=1.0, \frac{a}{t}=0.95$.

**A second example.**



# Another example of the high accuracy of $\Delta F_w$ for the case a/t=0.95.

| W/D | 1/7 | 1/6 | 1/5 | 1/4 | 1/3 | 1/2 | 2/3 | 3/4 | 4/5 | 1 | 5/4 | 4/3 | 3/2 | 2 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 10.0 | | 2.1 | 1.6 | 1.3 | 0.9 | 0.6 | 0.5 | 0.4 | 0.4 | 0.3 | 0.2 | 0.2 | 0.2 | 0.1 |
| 8.0 | | | | 2.1 | 1.6 | 1.0 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 | 0.4 | 0.3 | 0.2 |
| 7.0 | | | | 2.9 | 2.2 | 1.4 | 1.1 | 0.9 | 0.9 | 0.7 | 0.5 | 0.5 | 0.4 | 0.3 |
| 6.4 | | | | 3.7 | 2.7 | 1.8 | 1.3 | 1.1 | 1.1 | 0.8 | 0.6 | 0.6 | 0.5 | 0.4 |
| 5.8 | | | | 4.6 | 3.5 | 2.2 | 1.6 | 1.4 | 1.3 | 1.0 | 0.8 | 0.7 | 0.6 | 0.4 |
| 5.2 | | | | | 4.6 | 3.0 | 2.2 | 1.9 | 1.7 | 1.3 | 1.0 | 1.0 | 0.8 | 0.5 |
| 4.6 | | | | | | 4.1 | 2.9 | 2.5 | 2.4 | 1.8 | 1.4 | 1.3 | 1.1 | 0.7 |
| 4.0 | | | | | | 6.0 | 4.3 | 3.7 | 3.4 | 2.6 | 2.0 | 1.8 | 1.6 | 1.0 |
| 3.6 | | | | | | | 5.7 | 5.0 | 4.6 | 3.5 | 2.6 | 2.4 | 2.1 | 1.4 |
| 3.2 | | | | | | | | 7.1 | 6.5 | 4.9 | 3.7 | 3.4 | 2.9 | 1.9 |
| 2.8 | | | | | | | | | | 7.5 | 5.6 | 5.1 | 4.3 | 2.9 |
| 2.4 | | | | | | | | | | 12.6 | 9.3 | 8.5 | 7.2 | 4.8 |
| 2.2 | | | | | | | | | | | 12.8 | 11.7 | 9.9 | 6.5 |
| 2.0 | | | | | | | | | | | | | 14.3 | 9.4 |
| 1.8 | | | | | | | | | | | | | | 14.7 |

Errors are up to 14-15% !

**Error in % before applying the $\Delta F_w\left(\frac{W}{D},\frac{a}{c}\right)$-correction.**

| W/D | 1/7 | 1/6 | 1/5 | 1/4 | 1/3 | 1/2 | 2/3 | 3/4 | 4/5 | 1 | 5/4 | 4/3 | 3/2 | 2 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 10.0 | | 0.1 | 0.1 | 0.2 | 0.1 | 0.1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.1 | 0.0 | 0.1 | 0.1 |
| 8.0 | | | | 0.1 | 0.1 | 0.0 | 0.1 | 0.1 | 0.1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.2 |
| 7.0 | | | | 0.0 | 0.1 | 0.0 | 0.0 | 0.0 | 0.1 | 0.1 | 0.0 | 0.0 | 0.0 | 0.3 |
| 6.4 | | | | 0.0 | 0.1 | 0.1 | 0.0 | 0.1 | 0.0 | 0.1 | 0.1 | 0.1 | 0.0 | 0.0 |
| 5.8 | | | | 0.2 | 0.1 | 0.1 | 0.0 | 0.0 | 0.1 | 0.0 | 0.1 | 0.1 | 0.1 | 0.0 |
| 5.2 | | | | | 0.1 | 0.0 | 0.0 | 0.0 | 0.1 | 0.0 | 0.0 | 0.0 | 0.1 | 0.0 |
| 4.6 | | | | | | 0.1 | 0.0 | 0.1 | 0.1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.1 |
| 4.0 | | | | | | 0.1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 3.6 | | | | | | | 0.0 | 0.0 | 0.1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.1 |
| 3.2 | | | | | | | | 0.0 | 0.1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.1 |
| 2.8 | | | | | | | | | | 0.1 | 0.0 | 0.0 | 0.0 | 0.1 |
| 2.4 | | | | | | | | | | 0.0 | 0.1 | 0.1 | 0.1 | 0.1 |
| 2.2 | | | | | | | | | | | 0.1 | 0.1 | 0.0 | 0.1 |
| 2.0 | | | | | | | | | | | | | 0.1 | 0.1 |
| 1.8 | | | | | | | | | | | | | | 0.0 |

Errors are 0.0 - 0.2% !

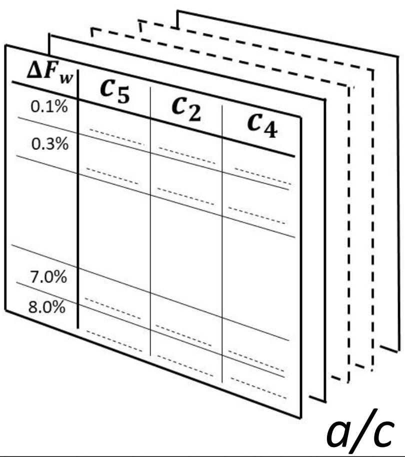
**Error in % after applying the $\Delta F_w\left(\frac{W}{D},\frac{a}{c}\right)$-correction.**

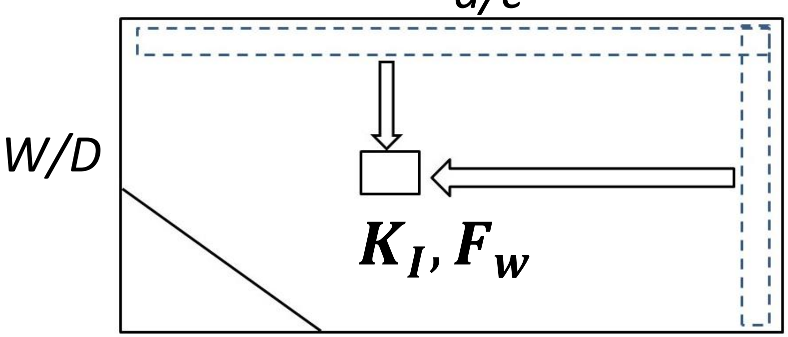
# Calculation of $\Delta F_w(\frac{W}{D}, \frac{a}{t}, \frac{a}{c})$ at vertex 'c' with high accuracy.

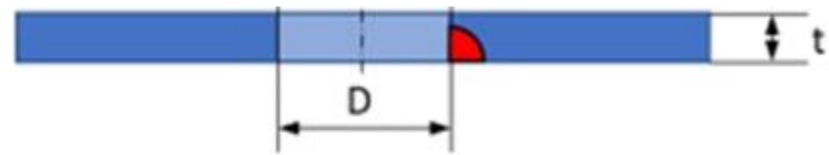
The equation,

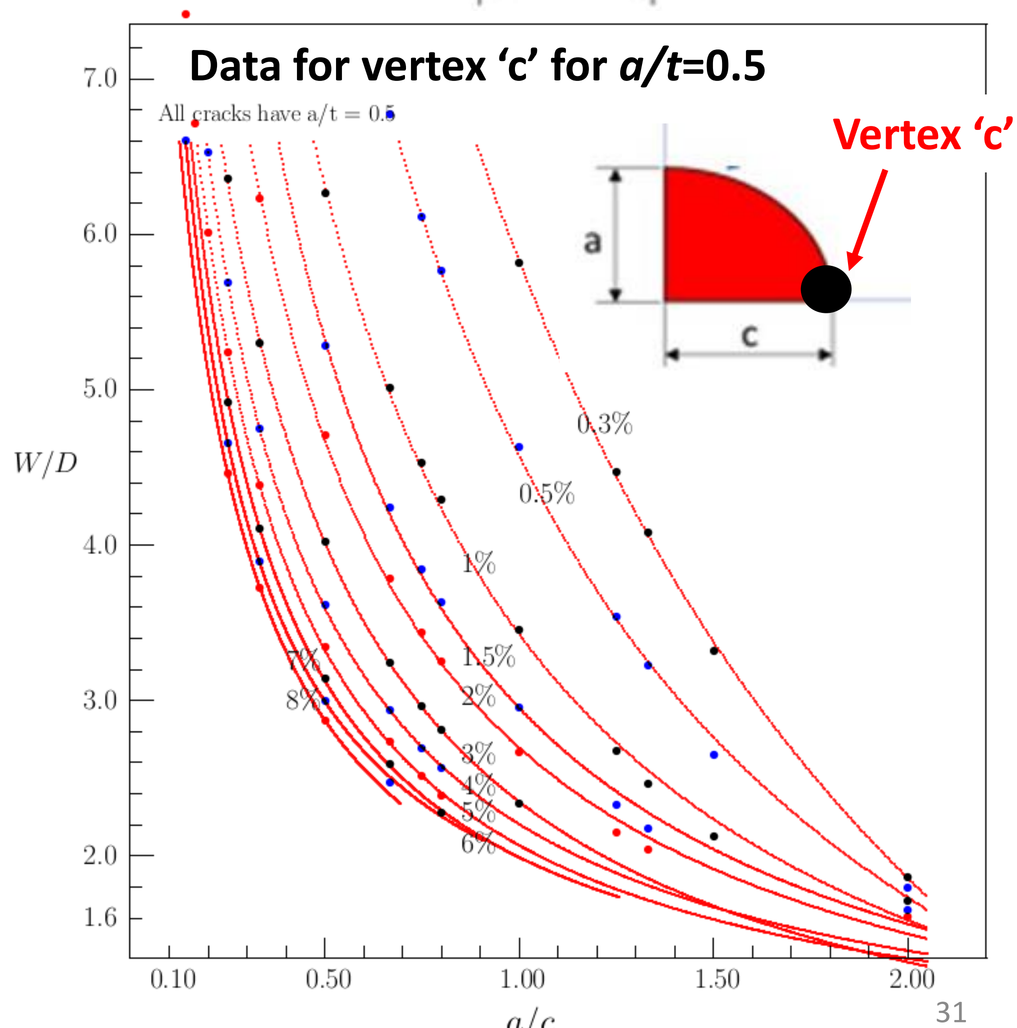
$$\frac{W}{D} = \frac{c_5(\Delta F_w)}{\frac{a}{c} - c_2(\Delta F_w)} + c_4(\Delta F_w)$$

approximates data in all tables very well.



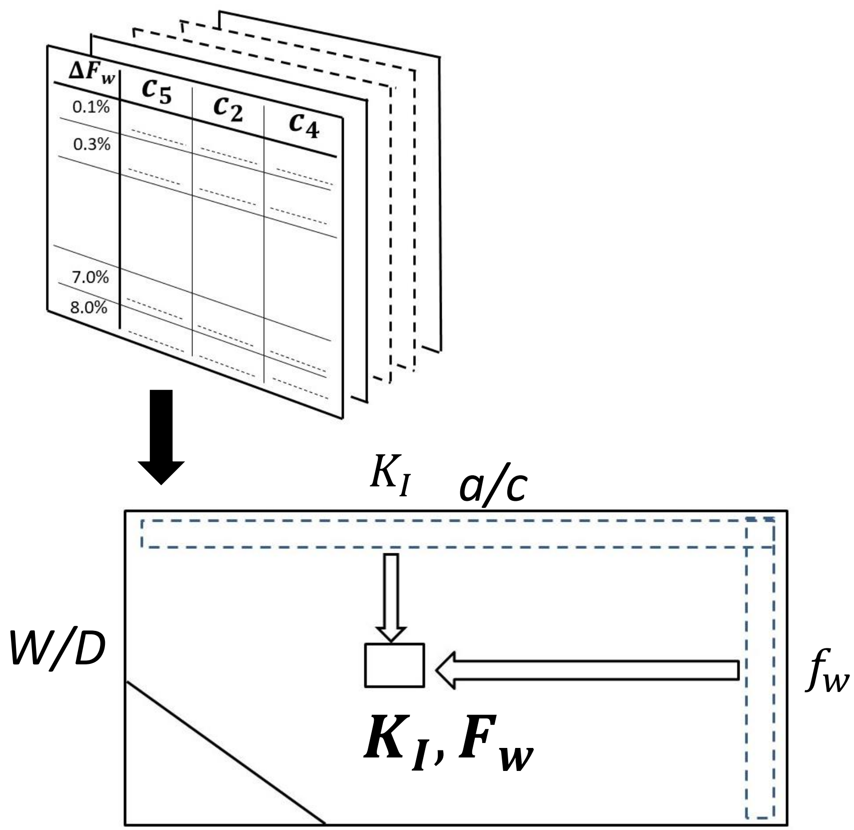



# We don't need functions $\Delta F_w\left(\frac{D}{t},\frac{W}{D},\frac{a}{t},\frac{a}{c}\right)$ with an error of order 0.1%?

**Accurate $F_w\left(\frac{D}{t},\frac{W}{D},\frac{a}{t},\frac{a}{c}\right)$-calculation.**

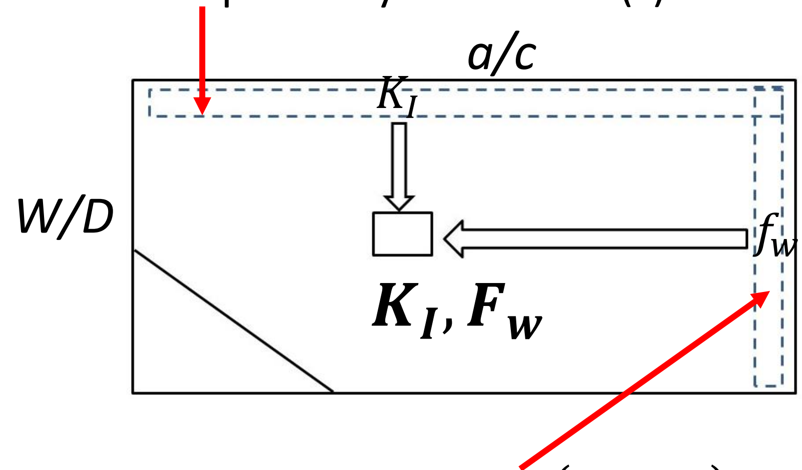
$$\frac{W}{D} = \frac{c_5(\Delta F_w)}{\frac{a}{c} - c_2(\Delta F_w)} + c_4(\Delta F_w)$$

**+**




**$F_w\left(\frac{D}{t},\frac{W}{D},\frac{a}{t},\frac{a}{c}\right)$-calculation with prescribed and known accuracy.**

**Step 1**: Approximate $K_I\left(\frac{W}{D}=\infty\right)$ with simple analytic function(s).




**Step 2**: Approximate $\mathrm{f_w}\left(\frac{a}{c}=10\right)$ with simple analytic function(s).

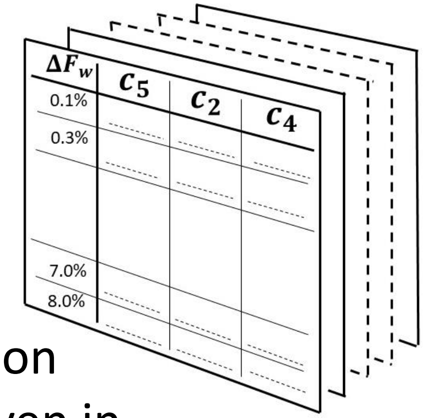
**Step 3**: Simplified $\Delta F_w\left(\frac{\mathrm{W}}{\mathrm{D}},\frac{\mathrm{a}}{\mathrm{t}},\frac{\mathrm{a}}{\mathrm{c}}\right)$ – calculation.

Total approximation error is product between the 3 errors and known!

# Step 3: More useful but slightly less accurate $\Delta F_w$-calculation.

**Summary: accurate $\Delta F_w$-calculation**

$$\frac{W}{D} = \frac{c_5(\Delta F_w)}{\frac{a}{c} - c_2(\Delta F_w)} + c_4(\Delta F_w) \quad + \quad \longrightarrow \quad \Delta F_w$$




A drawback in fatigue crack propagation analysis is that $\Delta F_w$ is not explicitly given in the equation above.

One (of several possible) useful approach is to replace table data with closed form analytic approximations of $c_5, c_2$ and $c_4$, respectively.
It was found that table data can be accurately approximated with the following functions (see example next slide).

$$c_5(\Delta F_w) \approx \frac{a_5}{\Delta F_w} + b_5, \quad c_2(\Delta F_w) \approx \frac{a_2}{\Delta F_w} + b_2 \text{ and } c_4(\Delta F_w) \approx \frac{a_4}{\Delta F_w} + b_4.$$

Where $a_i$ and $b_i$ are six constants for fixed *a/t* (or smooth functions of *a/t*).

## Cont. more useful but slightly less accurate $\Delta F_w$-calculation.

After inserting,

$$c_5(\Delta F_w) = \frac{a_5}{\Delta F_w} + b_5,\ \ c_2(\Delta F_w) = \frac{a_2}{\Delta F_w} + b_2 \text{ and } c_4(\Delta F_w) = \frac{a_4}{\Delta F_w} + b_4.$$

Into,

$$\frac{W}{D} = \frac{c_5(\Delta F_w)}{\frac{a}{c} - c_2(\Delta F_w)} + c_4(\Delta F_w)$$

We get,

$$\left(\frac{1}{\Delta F_w}\right)^2 + q \cdot \left(\frac{1}{\Delta F_w}\right) + r = 0$$

Where,

$$q = (-a_2 \cdot \frac{W}{D} - a_4 \cdot \frac{a}{c} - \mathrm{a}_5 + \mathrm{a}_2 \cdot b_4 + b_2 \cdot a_4)/(a_2 \cdot a_4)$$

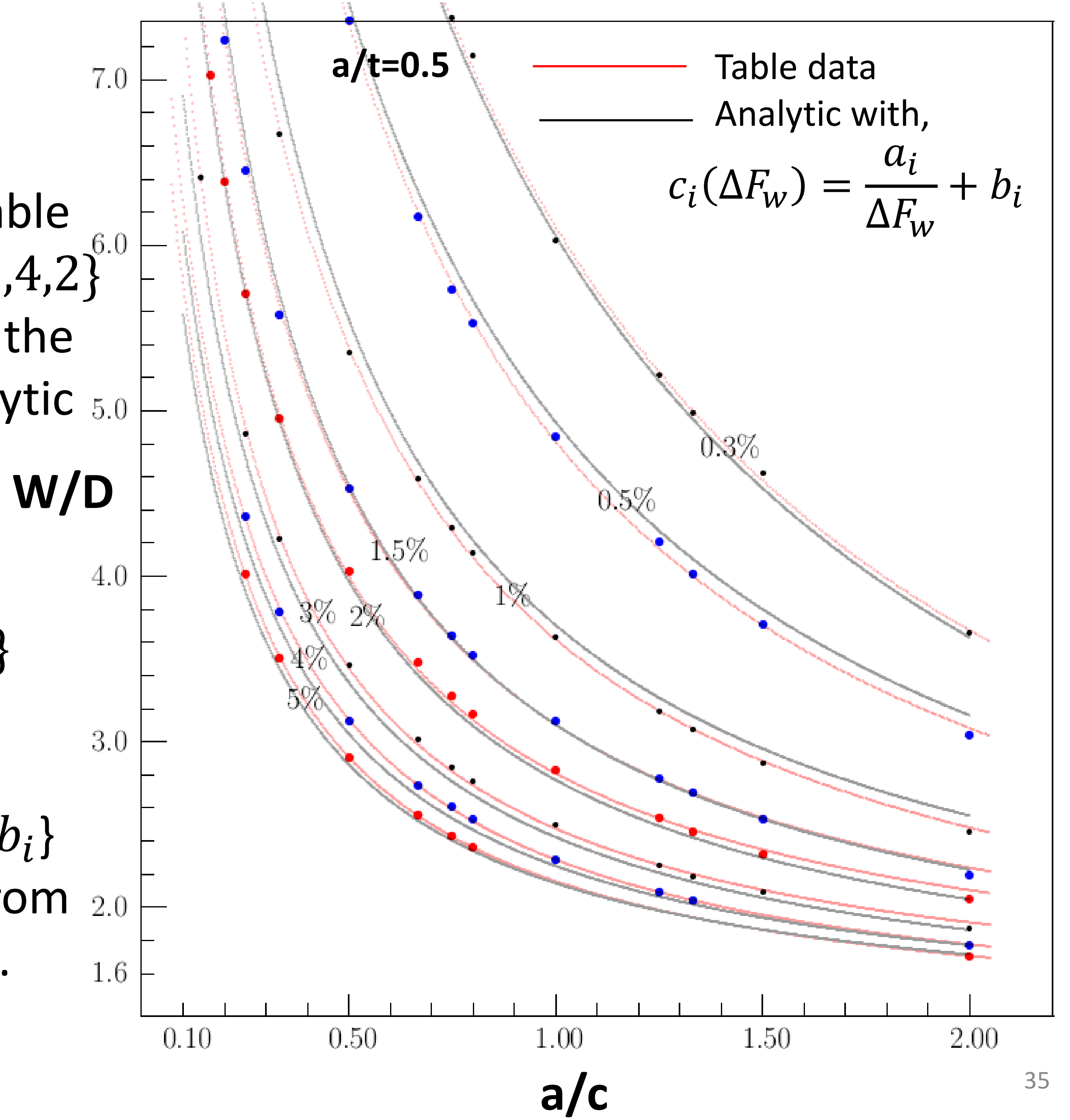
$$r = (\left(\frac{a}{c} - b_2\right) \cdot \frac{W}{D} - b_4 \cdot \frac{a}{c} - b5 + b_2 \cdot b_4)/(a_2 \cdot a_4)$$

With the closed form solution,

$$\boxed{\Delta F_w = 1/(-\frac{q}{2} - \sqrt{\left\{\frac{q^2}{4} - r\right\}})}$$

Graphs showing $\Delta F_w = constant$ obtained from table data of $\{c_i, i = 5,4,2\}$ (red curves) and the closed form analytic approximation using six coefficients $\{a_i, b_i, i = 5,4,2\}$ (black curves).

Coefficients $\{a_i, b_i\}$ were obtained from a least square fit.



# Cont. more useful but slightly less accurate $\Delta F_w$-calculation.

| W/D | 1/10 | 1/9 | 1/8 | 1/7 | 1/6 | 1/5 | 1/4 | 1/3 | 1/2 | 2/3 | 3/4 | 4/5 | 1 | 5/4 | 4/3 | 3/2 | 2 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 100.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 15.0 | 0.6 | 0.5 | 0.5 | 0.4 | 0.4 | 0.3 | 0.2 | 0.2 | 0.1 | 0.1 | 0.1 | 0.1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 12.0 | 1.0 | 0.9 | 0.8 | 0.7 | 0.6 | 0.5 | 0.4 | 0.3 | 0.2 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.0 |
| 10.0 | 1.6 | 1.4 | 1.2 | 1.1 | 0.9 | 0.7 | 0.6 | 0.4 | 0.3 | 0.2 | 0.2 | 0.2 | 0.1 | 0.1 | 0.1 | 0.1 | 0.0 |
| 8.0 | 2.7 | 2.4 | 2.1 | 1.8 | 1.5 | 1.2 | 0.9 | 0.7 | 0.4 | 0.3 | 0.2 | 0.2 | 0.2 | 0.1 | 0.1 | 0.1 | 0.1 |
| 7.0 | | | 2.9 | 2.4 | 2.0 | 1.6 | 1.3 | 0.9 | 0.6 | 0.4 | 0.3 | 0.3 | 0.2 | 0.2 | 0.2 | 0.1 | 0.1 |
| 6.4 | | | | 3.0 | 2.5 | 2.0 | 1.5 | 1.1 | 0.7 | 0.5 | 0.4 | 0.4 | 0.3 | 0.2 | 0.2 | 0.1 | 0.1 |
| 5.8 | | | | | | 2.5 | 1.9 | 1.4 | 0.8 | 0.6 | 0.5 | 0.4 | 0.3 | 0.2 | 0.2 | 0.2 | 0.1 |
| 5.2 | | | | | | | 2.5 | 1.8 | 1.1 | 0.7 | 0.6 | 0.6 | 0.4 | 0.3 | 0.3 | 0.2 | 0.1 |
| 4.6 | | | | | | | 3.5 | 2.4 | 1.4 | 1.0 | 0.8 | 0.8 | 0.6 | 0.4 | 0.4 | 0.3 | 0.2 |
| 4.0 | | | | | | | 5.0 | 3.4 | 2.0 | 1.4 | 1.2 | 1.1 | 0.8 | 0.6 | 0.5 | 0.4 | 0.2 |
| 3.6 | | | | | | | 6.9 | 4.6 | 2.7 | 1.8 | 1.5 | 1.4 | 1.0 | 0.7 | 0.7 | 0.5 | 0.3 |
| 3.2 | | | | | | | | 6.5 | 3.7 | 2.5 | 2.1 | 1.9 | 1.4 | 1.0 | 0.9 | 0.7 | 0.4 |
| 2.8 | | | | | | | | | | 3.7 | 3.1 | 2.9 | 2.1 | 1.5 | 1.3 | 1.1 | 0.6 |
| 2.4 | | | | | | | | | | 6.1 | 5.2 | 4.7 | 3.4 | 2.4 | 2.2 | 1.8 | 1.1 |
| 2.2 | | | | | | | | | | | | 6.4 | 4.6 | 3.3 | 2.9 | 2.4 | 1.5 |
| 2.0 | | | | | | | | | | | | | | 4.8 | 4.3 | 3.6 | 2.2 |
| 1.8 | | | | | | | | | | | | | | | | | 3.6 |
| 1.6 | | | | | | | | | | | | | | | | | 7.0 |

Errors are up to 6-7% !

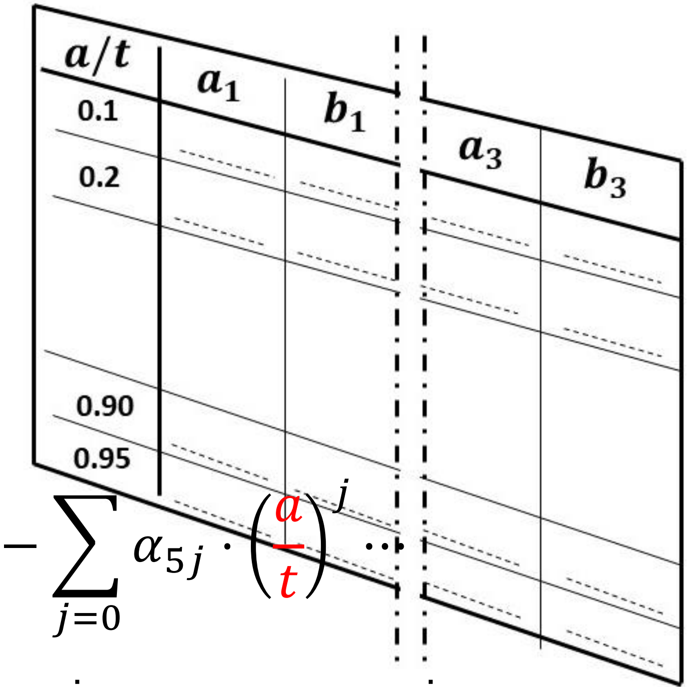
**Error in % before applying the $\Delta F_w\left(\frac{W}{D},\frac{a}{c}\right)$-correction.**

| W/D | 1/10 | 1/9 | 1/8 | 1/7 | 1/6 | 1/5 | 1/4 | 1/3 | 1/2 | 2/3 | 3/4 | 4/5 | 1 | 5/4 | 4/3 | 3/2 | 2 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 100.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 15.0 | 0.2 | 0.1 | 0.1 | 0.1 | 0.1 | 0.0 | 0.2 | 0.2 | 0.1 | 0.1 | 0.1 | 0.1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 12.0 | 0.3 | 0.2 | 0.2 | 0.1 | 0.1 | 0.0 | 0.0 | 0.3 | 0.2 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.0 |
| 10.0 | 0.4 | 0.3 | 0.2 | 0.2 | 0.1 | 0.0 | 0.0 | 0.0 | 0.3 | 0.2 | 0.2 | 0.2 | 0.1 | 0.1 | 0.1 | 0.1 | 0.0 |
| 8.0 | 0.7 | 0.5 | 0.4 | 0.3 | 0.2 | 0.1 | 0.0 | 0.0 | 0.0 | 0.3 | 0.2 | 0.2 | 0.2 | 0.1 | 0.1 | 0.1 | 0.1 |
| 7.0 | | | 0.6 | 0.4 | 0.3 | 0.2 | 0.1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.2 | 0.2 | 0.2 | 0.1 | 0.1 |
| 6.4 | | | | 0.4 | 0.3 | 0.2 | 0.1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.3 | 0.2 | 0.2 | 0.1 | 0.1 |
| 5.8 | | | | | | 0.2 | 0.2 | 0.1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.2 | 0.2 | 0.2 | 0.1 |
| 5.2 | | | | | | | 0.2 | 0.1 | 0.1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.3 | 0.2 | 0.1 |
| 4.6 | | | | | | | 0.3 | 0.2 | 0.1 | 0.1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.2 |
| 4.0 | | | | | | | 0.0 | 0.2 | 0.2 | 0.1 | 0.1 | 0.1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.2 |
| 3.6 | | | | | | | 1.2 | 0.0 | 0.3 | 0.2 | 0.2 | 0.2 | 0.1 | 0.0 | 0.0 | 0.0 | 0.0 |
| 3.2 | | | | | | | | 1.2 | 0.3 | 0.3 | 0.3 | 0.3 | 0.2 | 0.1 | 0.1 | 0.0 | 0.0 |
| 2.8 | | | | | | | | | 0.4 | 0.3 | 0.4 | 0.4 | 0.3 | 0.2 | 0.2 | 0.1 | 0.0 |
| 2.4 | | | | | | | | | | 1.1 | 0.2 | 0.0 | 0.3 | 0.3 | 0.3 | 0.2 | 0.1 |
| 2.2 | | | | | | | | | | | | 1.4 | 0.0 | 0.3 | 0.3 | 0.3 | 0.2 |
| 2.0 | | | | | | | | | | | | | | 0.3 | 0.0 | 0.2 | 0.3 |
| 1.8 | | | | | | | | | | | | | | | | | 0.1 |
| 1.6 | | | | | | | | | | | | | | | | | 1.4 |

Errors are 0.0 – 0.3% except for the largest cracks where errors reach 1.4%.

**Error in % after applying the approximative $\Delta F_w\left(\frac{W}{D},\frac{a}{c}\right)$-correction.**

$$c_i(\Delta F_w) = \frac{a_i(\frac{a}{t})}{\Delta F_w} + b_i(\frac{a}{t})$$, see Table.




Data in each column can be approximated accurately with,

$$a_i \approx \sum_{j=0} \alpha_{ij} \cdot \left(\frac{a}{t}\right)^{\mathrm{j}} \qquad b_i \approx \sum_{j=0} \beta_{ij} \cdot \left(\frac{a}{t}\right)^{\mathrm{j}}$$

$$q = (-\frac{W}{D} \cdot \sum_{j=0} \alpha_{2j} \cdot \left(\frac{a}{t}\right)^{j} - \frac{a}{c} \cdot \sum_{j=0} \alpha_{4j} \cdot \left(\frac{a}{t}\right)^{j} - \sum_{j=0} \alpha_{5j} \cdot \left(\frac{a}{t}\right)^{j} \cdots$$

$$r = (\frac{W}{D} \cdot \left(\frac{a}{c} - \sum_{j=0} \beta_{2j} \cdot \left(\frac{a}{t}\right)^{\mathrm{j}}\right) - \frac{a}{c} \cdot \sum_{j=0} \beta_{4j} \cdot \left(\frac{a}{t}\right)^{\mathrm{j}} - \sum_{j=0} \beta_{5j} \cdot \left(\frac{a}{t}\right)^{\mathrm{j}}$$

$$\Delta F_w(\frac{W}{D}, \frac{a}{t}, \frac{a}{c}, \underbrace{\alpha_{ij}, \beta_{ij}}_{\approx 20 \text{ coefficients}}) = 1/(-\frac{q(\cdots)}{2} - \sqrt{\left\{\frac{q(\cdots)^2}{4} - r(\cdots)\right\}})$$

**Verification: Average error in $\Delta F_w$ is 0.2% in 4400 control points (*R*/*t*=1).**

Next step: Determination of $\alpha_{ij}\left(\frac{R}{t}\right)$ and $\beta_{ij}\left(\frac{R}{t}\right)$.

**A few remarks on how to (perhaps) use the finite width correction function above in AFGROW, NASGRO, FASTRAN.**

# AFGROW usage: $\Delta F_w$ for one domain with fixed $W/D$

We had,

$$\left(\frac{1}{\Delta F_w}\right)^2 + q \cdot \left(\frac{1}{\Delta F_w}\right) + r = 0$$

For *W/D* fixed (one specimen) we get,

$$q = (-a_2 \cdot \frac{W}{D} - a_4 \cdot \frac{a}{c} - \mathrm{a}_5 + \mathrm{a}_2 \cdot b_4 + b_2 \cdot a_4)/(a_2 \cdot a_4)$$

$$r = ((\frac{a}{c} - b_2) \cdot \frac{W}{D} - b_4 \cdot \frac{a}{c} - b5 + b_2 \cdot b_4)/(a_2 \cdot a_4)$$

Where $Q_i$ and $R_i$ are constants dependent on *W/D* and *R/t.*

$$q = Q_1 \cdot \frac{a}{c} + Q_2$$

$$r = R_1 \cdot \frac{a}{c} + R_2$$

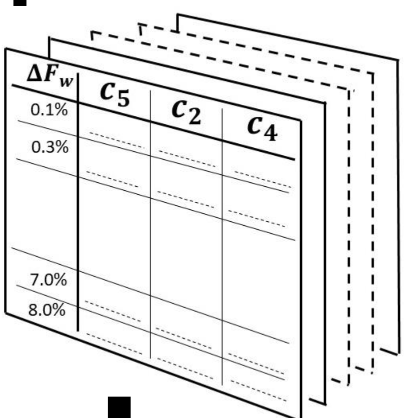
$$\Delta F_w = 1/(-\frac{q}{2} - \sqrt{\left\{\frac{q^2}{4} - r\right\}})$$

# AFGROW usage, continued.

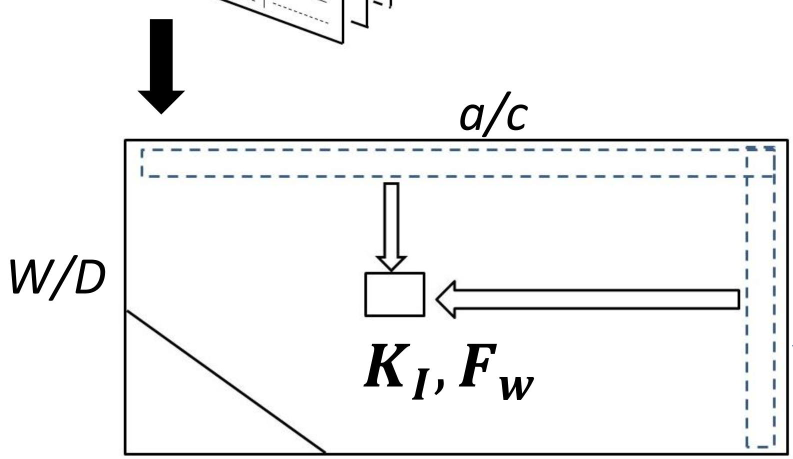
**Accurate $F_w\left(\frac{D}{t},\frac{W}{D},\frac{a}{t},\frac{a}{c}\right)$-calculation.**

$$\frac{W}{D}=\frac{c_5(\Delta F_w)}{\frac{a}{c}-c_2(\Delta F_w)}+c_4(\Delta F_w)$$

**+**

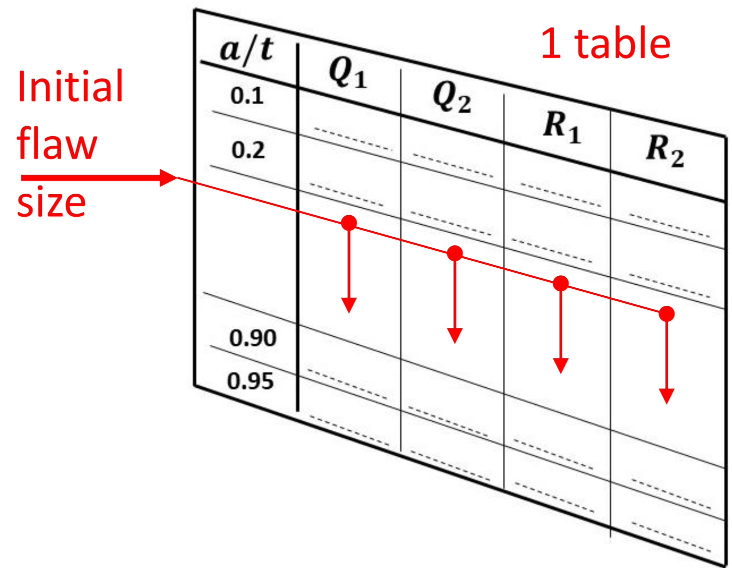



10 tables for various **a/t**.




**$F_w\left(\frac{D}{t},\frac{W}{D},\frac{a}{t},\frac{a}{c}\right)$-calculation with prescribed and known accuracy.**

$$c_i(\Delta F_w)=\frac{a_i}{\Delta F_w}+b_i.$$




$$q=Q_1\cdot\frac{a}{c}+Q_2,\; r=R_1\cdot\frac{a}{c}+R_2.$$

$$\Delta F_w=1/\left(-\frac{q}{2}-\sqrt{\left\{\frac{q^2}{4}-r\right\}}\right)$$

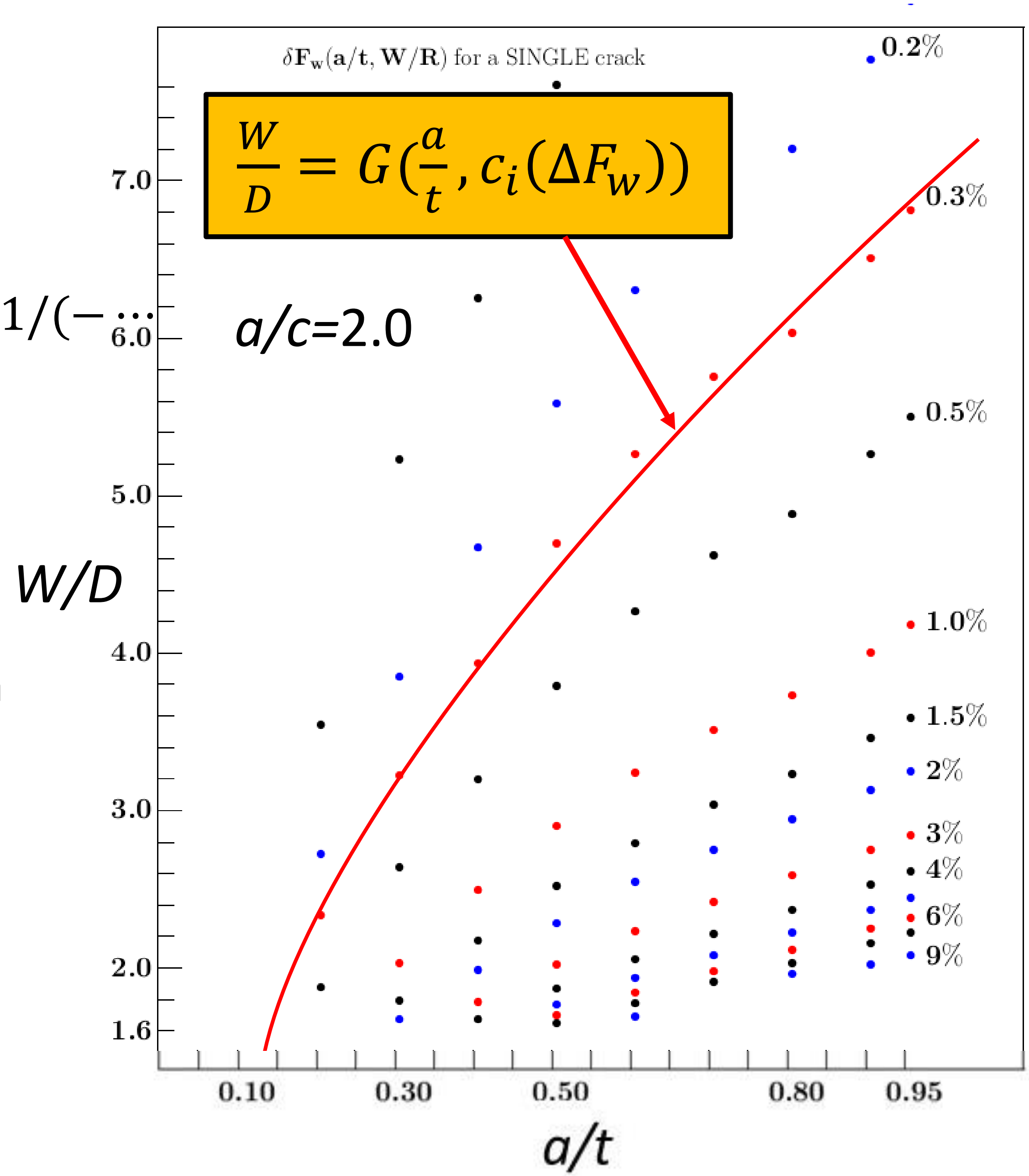
# Are significant improvements possible?

We have,

$$\Delta F_w\left(\frac{W}{D},\frac{a}{t},\frac{a}{c},\alpha_{ij},\beta_{ij}\right) = 1/(-\ldots$$

Challenge:

Find function $G(\frac{a}{t}, c_i)$ which is in close agreement with all data points for all fixed *a/c*-values so the $\alpha_{ij},\beta_{ij}$-dependence can be reduced/removed.

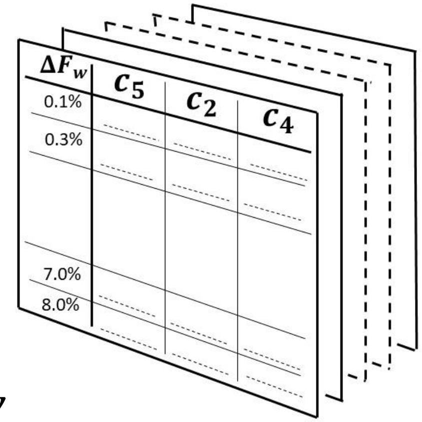


# Concluding remarks

1. **Higly accurate $\Delta F_w$-calculation**

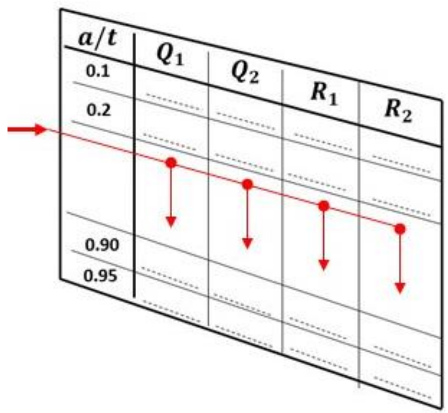
$$\frac{W}{D} = \frac{c_5(\Delta F_w)}{\frac{a}{c} - c_2(\Delta F_w)} + c_4(\Delta F_w)$$

+




→ $\Delta F_w$

2. **A closed form expression for $\Delta F_w$**




+

$$q = Q_1 \cdot \frac{a}{c} + Q_2, \; r = R_1 \cdot \frac{a}{c} + R_2.$$

↓

$$\Delta F_w = 1/(-\frac{q}{2} - \sqrt{\left\{\frac{q^2}{4} - r\right\}})$$

3. **A general closed form expression for $\Delta F_w$ for $D/t$ fixed.**

$$\Delta F_w(\frac{W}{D}, \frac{a}{t}, \frac{a}{c}, \alpha_{ij}, \beta_{ij}) = 1/(-\frac{q(\cdots)}{2} - \sqrt{\left\{\frac{q(\cdots)^2}{4} - r(\cdots)\right\}})$$

4. **The general closed form expression for $\Delta F_w$**

$$\Delta F_w\left(\frac{D}{t}, \frac{W}{D}, \frac{a}{t}, \frac{a}{c}, \epsilon_{ij}, \gamma_{ij}\right) = ???$$

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