Mechanical and Aerospace Engineering
University of California, Davis

2016 AFGROW Workshop:

Weight functions for a finite width plate with a radial crack at a circular hole

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Effects of RS on fracture

- **Where does RS appear in the BVP ingredient list?**
  - Strain-displacement equations
  - Stress equilibrium equations
  - Strain compatibility equations
  - Hooke's law (constitution)
  - Traction boundary conditions (Cauchy)
  - Displacement boundary conditions (support conditions)
- **Superposition approaches are typical**
  - Linear elastic – superposition of stress intensity factor
    - $K_{Tot} = K_{App} + K_{RS}$
  - Elastic-plastic (non-linear superposition)
    - Secondary load
    - Fix up for interaction effects
  - Non-linear analysis – include RS an initial condition

Effects of RS on fracture

- **Where does RS appear in the BVP ingredient list?**
  - Hooke's law (constitution)

    $\underline{\sigma} = \underline{\underline{\mathbf{C}}} : (\underline{\varepsilon} - \underline{\varepsilon}^0) + \underline{\sigma}^0$, where

    $\underline{\varepsilon}^0$ is initial strain

    $\underline{\sigma}^0$ is initial stress
    - Most commercial FE codes allow initial strain and/or stress, but may not handle both equally well
- **Superposition approaches are typical**
  - Linear elastic – superposition of stress intensity factor
    - $K_{Tot} = K_{App} + K_{RS}$
  - Elastic-plastic (non-linear superposition)
    - Secondary load
    - Fix up for interaction effects
  - Non-linear analysis – include RS an initial condition

Superposition for residual stress

- **Elastic model a useful starting point**
  - $K_{tot} = K_{app} + K_{rs}$
- **Clever superposition means we only need to know RS on the crack line, in the uncracked body**

Ref: H. Bueckner (1958) "The propagation of cracks and the energy of elastic deformation," Trans ASME, 80:1225–1230.

- **In EPFM, use K_{rs} as a starting point and "secondary load"**
  - Assumed not to drive plasticity
  - $J_{tot} = J_{el} + J_{pl}$
  - $J_{tot} = (K_{rs}+K_{app})^2/E' + J_{pl,app}$
- **Otherwise, full non-linear analysis, using full-field RS as an initial condition**
  - Difficult, but possible
    - Process simulation
    - Eigenstrain methods

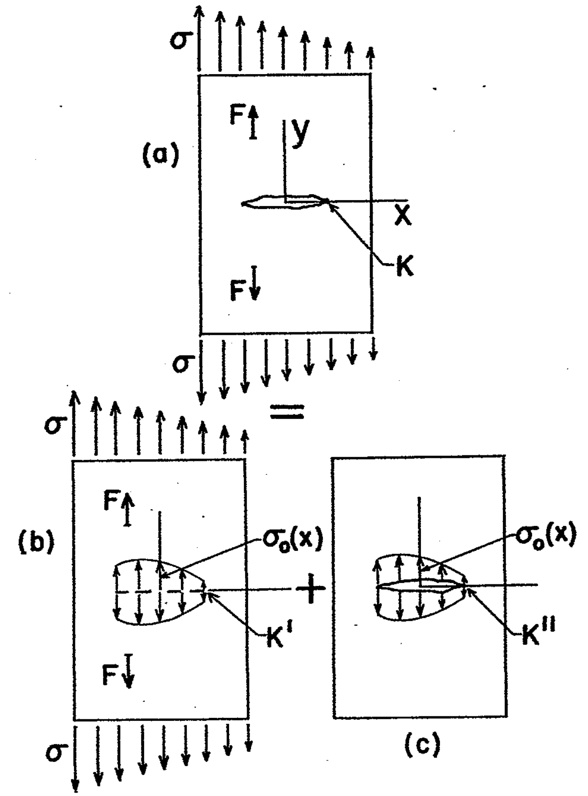
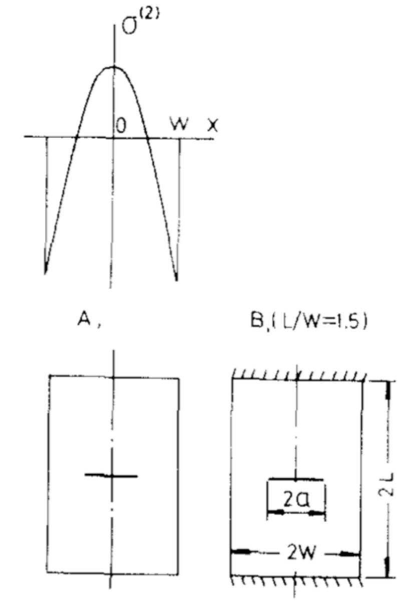


FIG. 6. THE REDUCTION OF A PROBLEM, (a), INTO TWO SIMPLER PROBLEMS, (b) and (c), FOR COMPUTATIONS OF STRESS SINGULARITY - INTENSITY FACTORS

Fig Ref: P.C. Paris, M.P. Gomez and W.E. Anderson, "A rational analytic theory of fatigue," The Trend in Engineering, 13, 9-14 (1961), Univ. of Washington, Seattle, WA.

Weight-function methods: Concept

- **Allows calculation of stress intensity factor for:**
  - *Arbitrary* crack-line stress distribution
  - *Specific* geometry
  - *Specific* set of boundary conditions



- **$\sigma^{(2)}(x)$ = stress field of interest**
  $K^{(2)}(a)$ = desired stress intensity factor

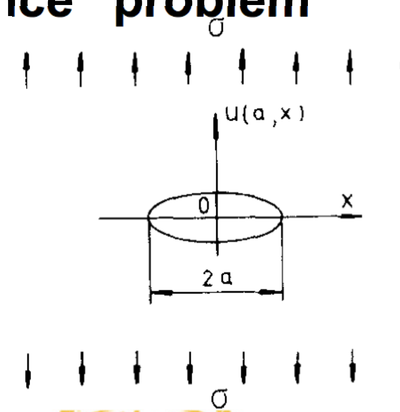
$$K^{(2)}(a) = \int_0^a \sigma^{(2)}(x) \cdot m(a,x)\, dx$$

m(a,x) = the weight function

- **The weight function is derived via a simpler "reference" problem**

$$m(a,x) = \frac{E'}{K^{(1)}} \frac{\partial u^{(1)}}{\partial a}$$

where
$K^{(1)}(a)$ = the reference stress intensity factor, and
$u^{(1)}(a,x)$ = the reference crack opening shape



Ref: XR Wu, Eng Fract Mech v20, 1984

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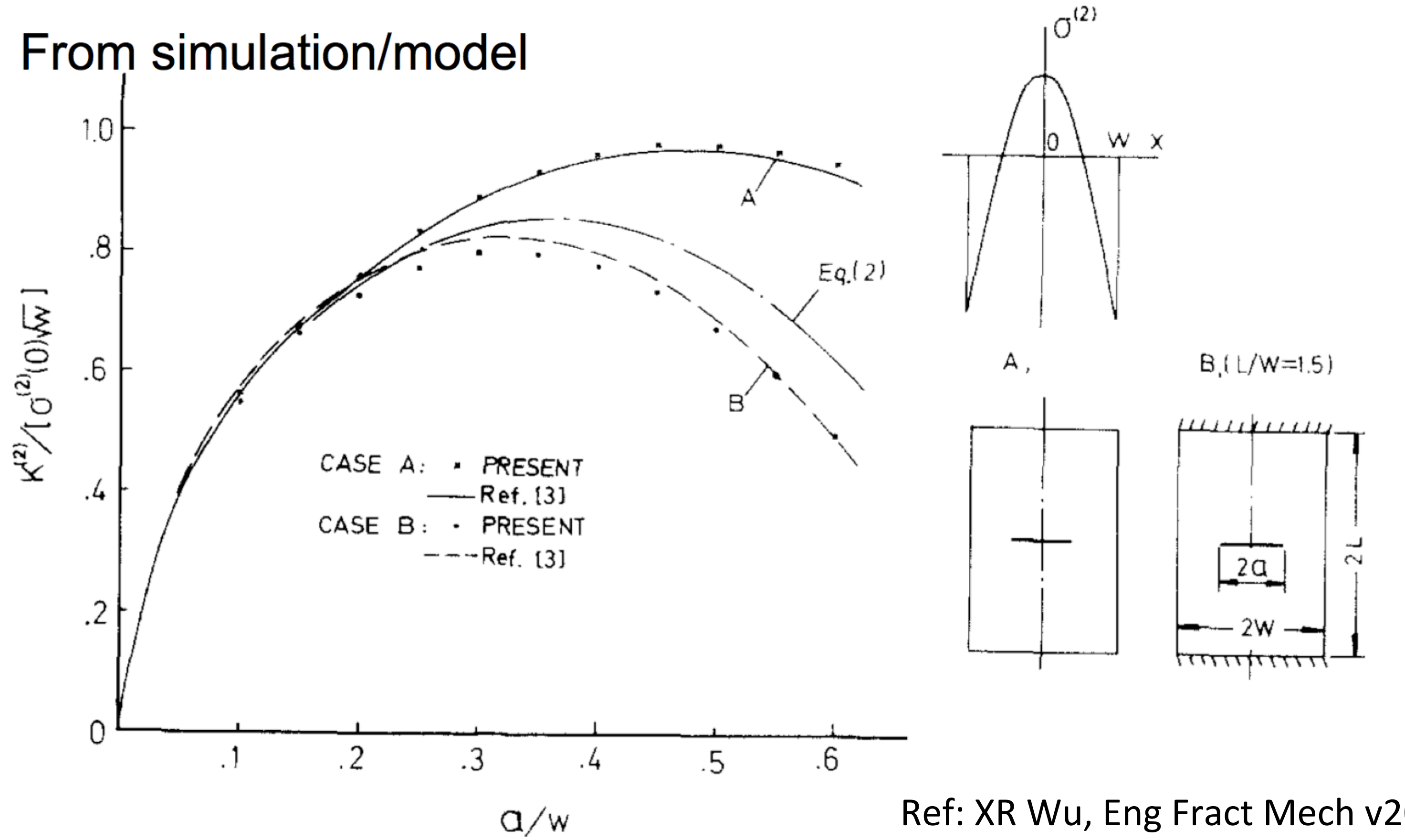
Weight-function methods: Residual stress applications

- **Use crack-line opening stresses in uncracked body**
  - From measurement
  - From simulation/model



Ref: XR Wu, Eng Fract Mech v20, 1984

Fig. 5. $K^{(2)}$-factors for a center crack in a plate, calculated using different weight functions. ($\sigma^{(2)}(x)$ is defined by eqn 26.)

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Weight-function methods: Take care with BCs

- **Must account for boundary conditions**
  - Use wrong BCs, get wrong answer
- **Problems largest for**
  - Edge cracks
  - Non-equilibruim stress states (esp. bending)
  - Constraints near to crack faces
- **Example**
  - Geom: Edge cracked strip
  - BC: Two different cases
    —— Free boundaries, or
    ---- Clamped sides
  - Stress: 3 different fields

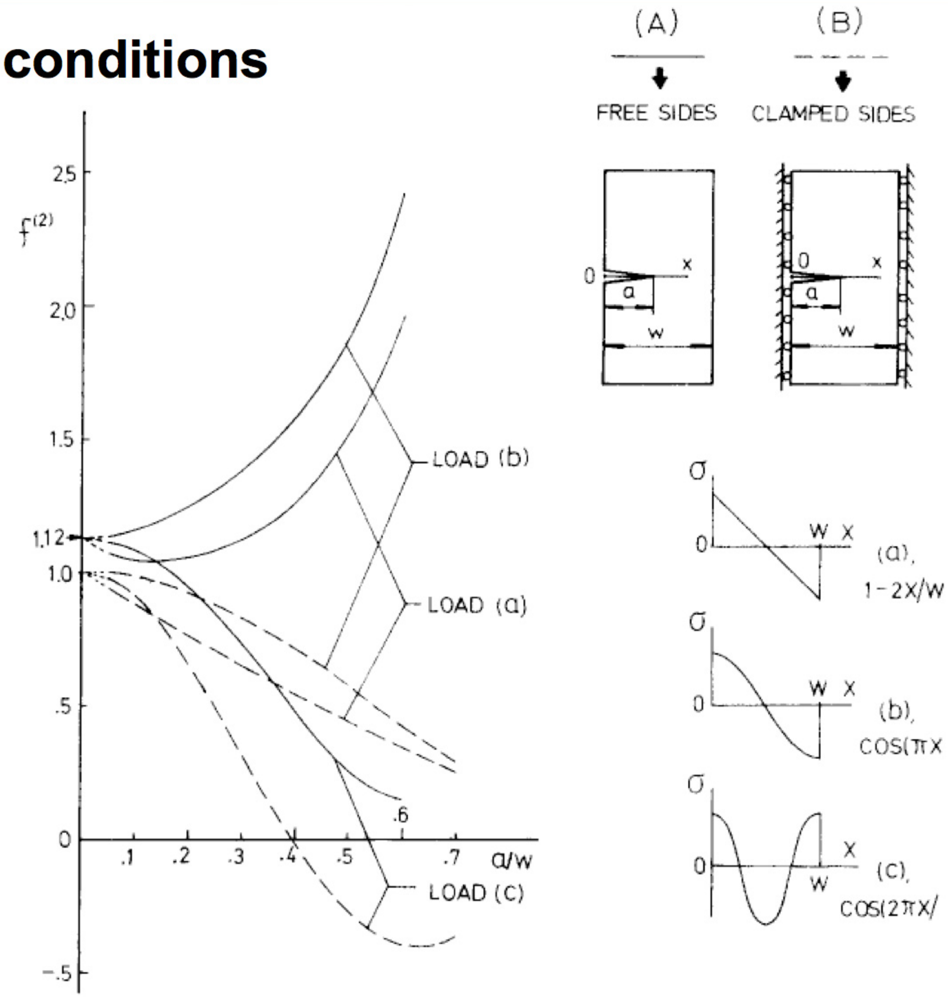
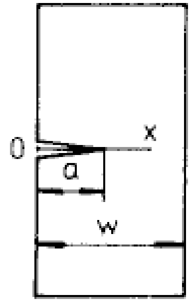


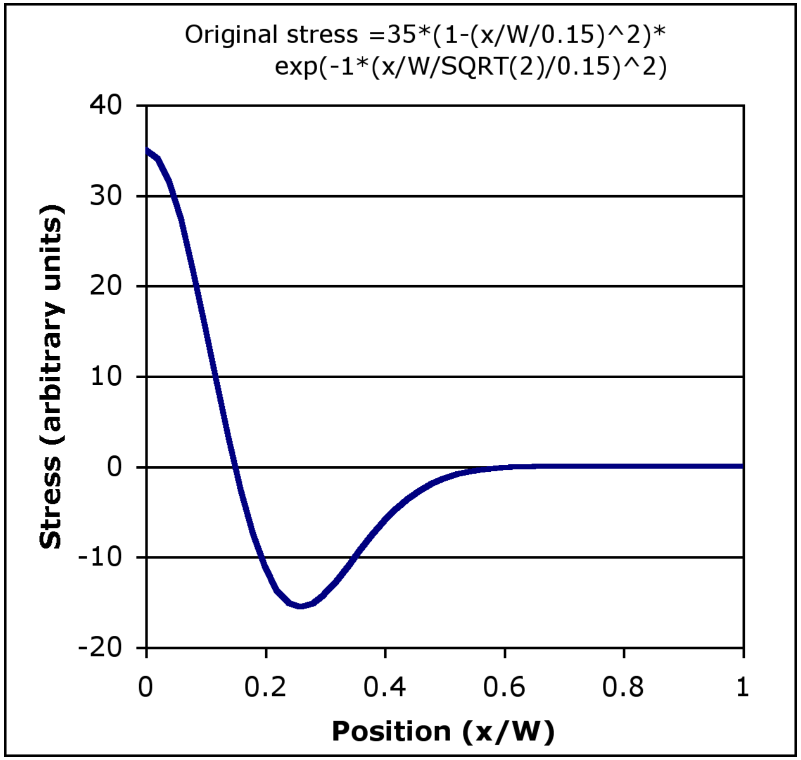
Fig. 8. $f^{(2)}$-factors for an edge crack subjected to various crack line stresses in: (A), unconstrained plate; (B), laterally constrained plate.

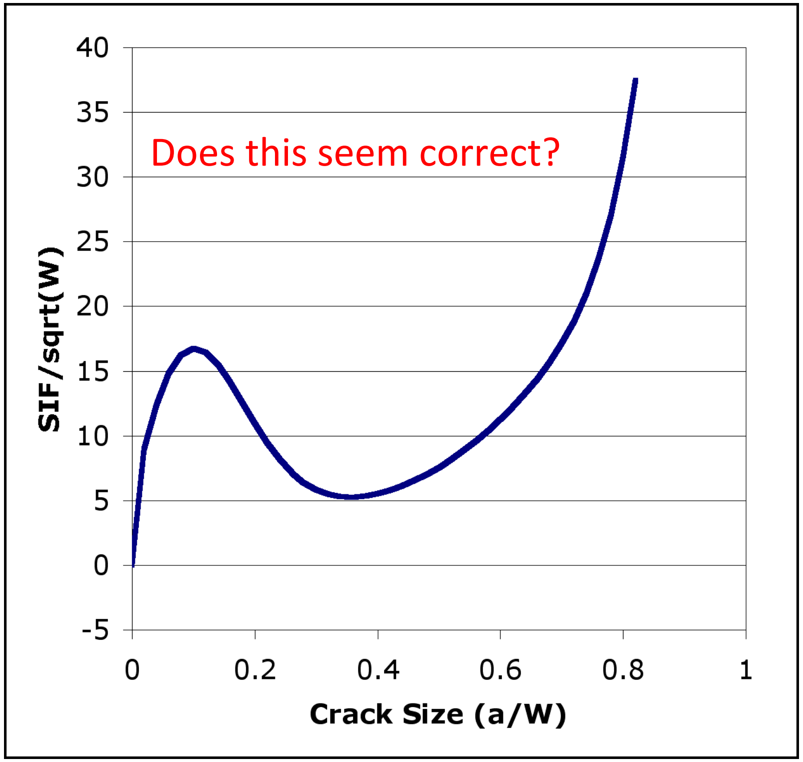
Ref: XR Wu, Eng Fract Mech v20, 1984

Weight-function methods: Take care with stress input

- **Residual stresses satisfy equilibrium**
  - $\sigma^{(2)}(x)$ must satisfy equilibrium to be useful
  - Small differences in $\sigma^{(2)}(x)$ can make large differences in $K^{(2)}(a)$
- **Example for edge-cracked strip**







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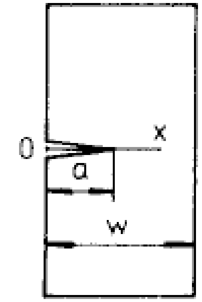

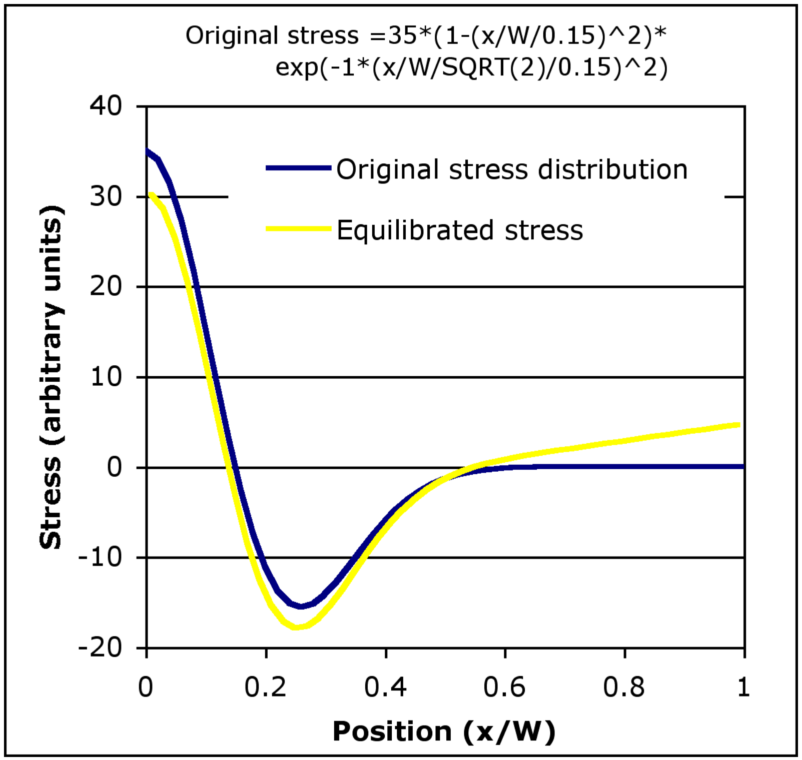

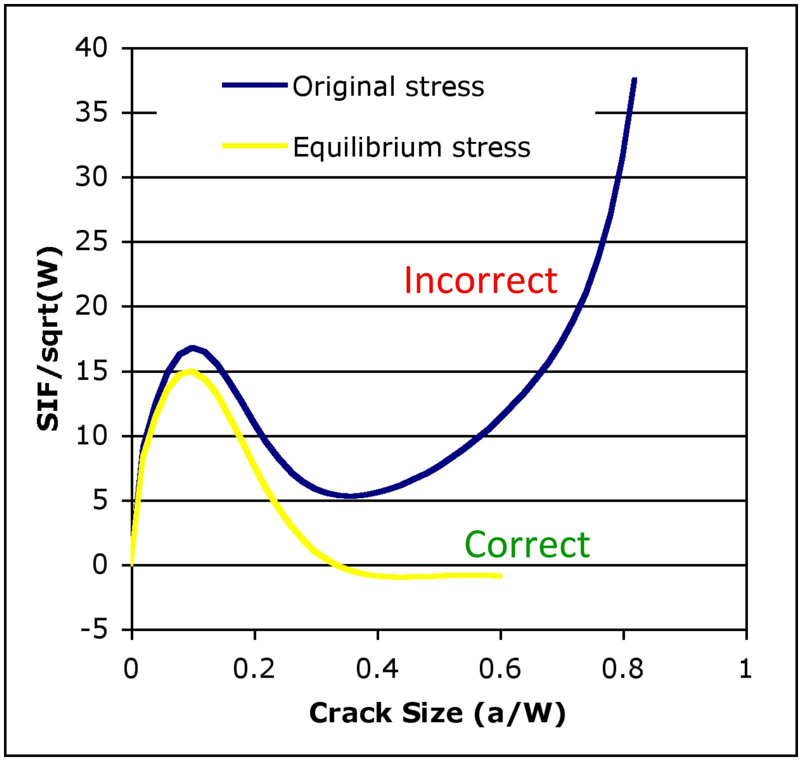

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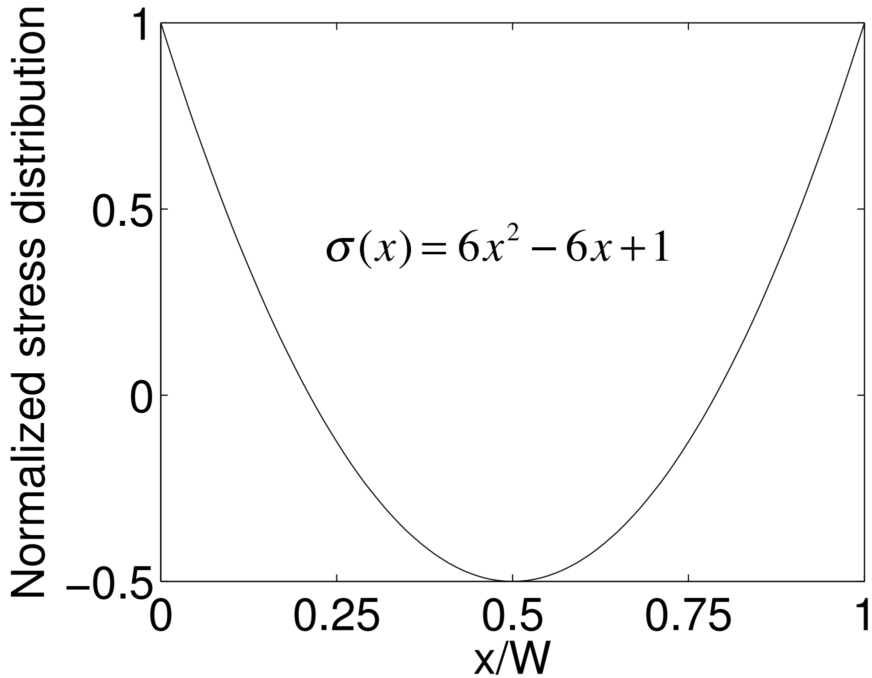
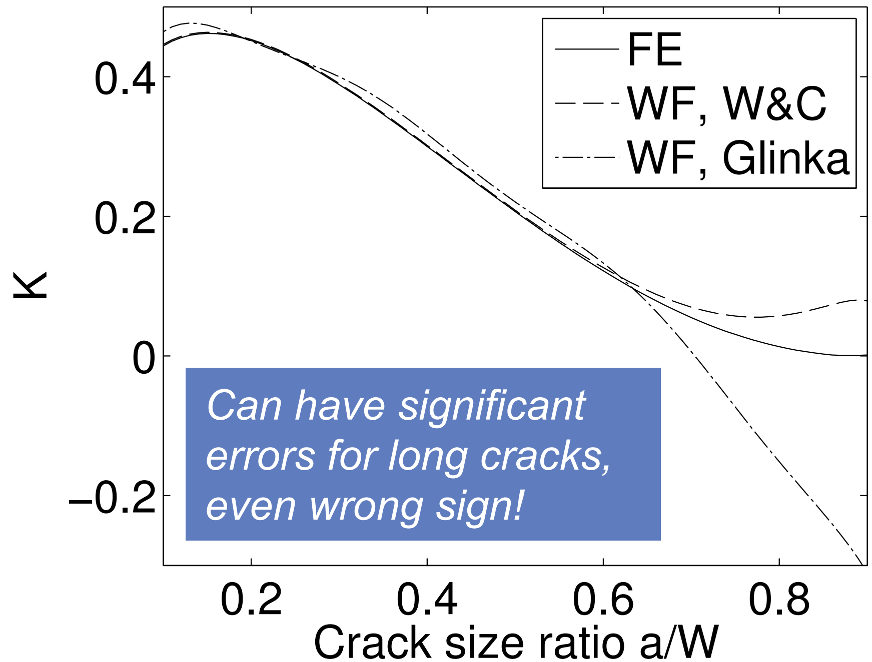
Weight-function methods: Some advice

- **Use correct Weight Function**
  - Match geometry
  - Match boundary conditions
- **Available in books and archival publications**
  - Example:
    Wu, XR and J Carlsson, 1991, Weight functions and stress intensity factor solutions, 1st ed, New York: Pergamon Press
- **Use residual stresses that satisfy equilibrium**
  - Experimental data often do not satisfy equilibrium and *must be* adjusted (basic requirement: satisfy equilibrium)
- **Benchmark analysis methods against available/reliable solutions and data**

Caution regarding published weight functions

- **Quality of available weight functions is variable**
  - Recommend benchmarking against high-quality solutions
- **Example: Use two different weight functions to compute SIF in edge-cracked strip for 2nd order Legendre stress (i.e., a very simple equilibrium stress field)**
  - Both weight functions "accurate to 1%", but only relative to simple loadings from which they were derived (uniform stress, linear stress)





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Wu and Carlsson method for deriving a weight function

- **Wu and Carlsson devised a simplified method for finding m(a,x) from a reference solution having only SIF and crack line stress**

$$f_r(a/W) = \frac{K_r(a/W)}{\sigma\sqrt{\pi a}}$$

$$\frac{\sigma_r(x/W)}{\sigma} = \sum_{m=0}^{M} S_m \left(\frac{x}{W}\right)^m$$

- **For edge cracks:**

$$m(a,x) = \frac{1}{\sqrt{2\pi a}} \sum_{i=1}^{3} \beta_i(a/W) \cdot \left(1 - \frac{x}{a}\right)^{i-\frac{3}{2}}$$

Ref: Wu, XR and J Carlsson, 1991, Weight functions and stress intensity factor solutions

Weight Functions
and
Stress Intensity Factor Solutions

XUE-REN WU
Institute of Aeronautical Materials, Beijing, China

and

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Royal Institute of Technology Stockholm, Sweden

PERGAMON PRESS

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Wu and Carlsson method for deriving a weight function

- **For edge cracks:**

Ref: Wu, XR and J Carlsson, 1991, Weight functions and stress intensity factor solutions

$$m(a,x) = \frac{1}{\sqrt{2\pi a}} \sum_{i=1}^{3} \beta_i(a/W) \cdot \left(1 - \frac{x}{a}\right)^{i-\frac{3}{2}}$$

$$f_r(a/W) = \frac{K_r(a/W)}{\sigma\sqrt{\pi a}}$$

$$\frac{\sigma_r(x/W)}{\sigma} = \sum_{m=0}^{M} S_m \left(\frac{x}{W}\right)^m$$

$$\beta_1(a/W) = 2.0$$

$$\beta_2(a/W) = \frac{1}{f_r(a/W)}\left[\frac{4a}{W} f_r'(a/W) + 2f_r(a/W) + \frac{3}{2}F_2(a/W)\right]$$

$$\beta_3(a/W) = \frac{1}{f_r(a/W)}\left[\frac{a}{W} F_2'(a/W) - \frac{1}{2}F_2(a/W)\right]$$

Wu and Carlsson method for deriving a weight function

- **For edge cracks:**

Ref: Wu, XR and J Carlsson, 1991, Weight functions and stress intensity factor solutions

$$F_1(a/W) = 4f_r(a/W)$$

$$F_2(a/W) = \frac{1}{E_2(a/W)}\left[\sqrt{2}\pi\phi(a/W) - E_1(a/W)\cdot F_1(a/W)\right]$$

$$\phi(a/W) = \frac{1}{(a/W)^2}\int_0^{a/W} s\cdot[f_r(s)]^2 ds$$

$$f_r(a/W) = \frac{K_r(a/W)}{\sigma\sqrt{\pi a}}$$

$$\phi'\left(\frac{a}{W}\right) = -\frac{2}{a/W}\phi(a/W) + \frac{1}{a/W}[f_r(a/W)]^2$$

$$\frac{\sigma_r(x/W)}{\sigma} = \sum_{m=0}^{M} S_m\left(\frac{x}{W}\right)^m$$

$$E_j(a/W) = \sum_{m=0}^{M}\frac{2^{m+1}m!\,S_m(a/W)^m}{\prod_{k=0}^{m}(1+2j+2k)}$$

- **When M = 0 (uniform crack face load)**

$$E_1(a/W) = \frac{2}{3} \text{ and } E_2(a/W) = \frac{2}{5}$$

Wu and Carlsson weight function for strip with center hole

- **Reference SIF and stress from earlier work**
  - Remote applied gross stress
  - Single crack: 2B/D = B/R = 2, 2.5, 3.5, 5
  - Double crack: 2B/D = B/R = 2, 4
- **Note: W = B – R**
  W = ligament, not specimen width
- **Challenges:**
  - Limited range of geometry
    Our past tests had B/R ≈ 6
  - Reference case uses
    applied gross stress, which limits
    accuracy of the weight function
    - Large value of M to fit stress field
      complicates distributions of $E_j(a/W)$

$$f_r(a/W) = \frac{K_r(a/W)}{\sigma\sqrt{\pi a}}$$

$$\frac{\sigma_r(x/W)}{\sigma} = \sum_{m=0}^{M} S_m \left(\frac{x}{W}\right)^m$$

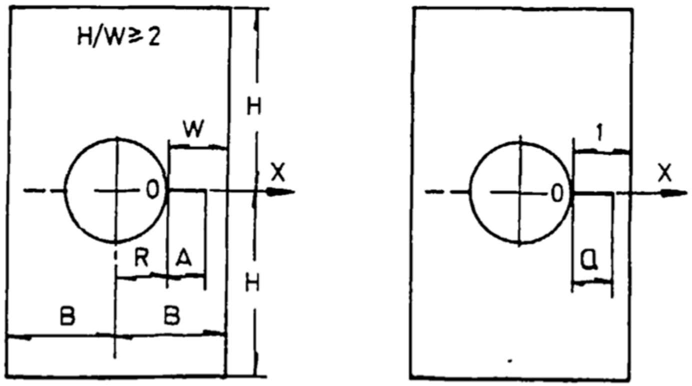
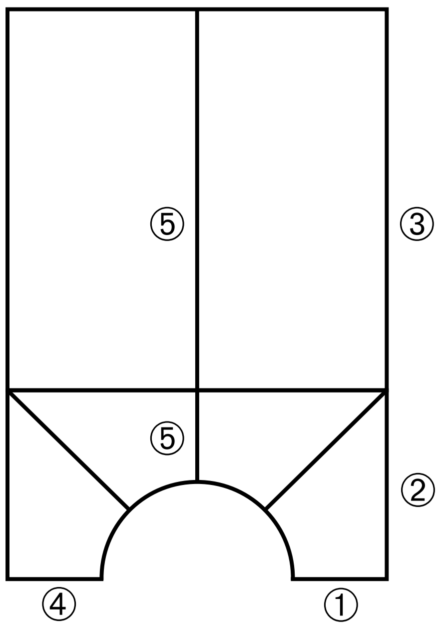


Fig. 14.1. Radial crack(s), $N = 1, 2$, at a circular hole in a finite plate.

New weight function for a strip with a center hole

- **Develop new reference SIF solutions using finite element methods with highly refined meshes and converged SIF results**
- **Long strip, central hole**
  - B/R = 2, 2.5, 3, 4, 6, and 10
  - Uniform crack line stress
  - Single crack
    - $0 \le a/W \le 0.9$
  - Double crack
    - $0 \le a/W \le 0.9$
- **Square plate (lug), central hole**
- **Fit SIF solutions with B-splines**
- **Develop tabular results for future use**
  - $\beta_1(a/W)$, $\beta_2(a/W)$, $\beta_3(a/W)$
  - $$m(a,x) = \frac{1}{\sqrt{2\pi a}} \sum_{i=1}^{3} \beta_i(a/W) \cdot \left(1 - \frac{x}{a}\right)^{i-\frac{3}{2}}$$



New weight function for a strip with a center hole

- Tabulated spline fits to reference normalized SIF

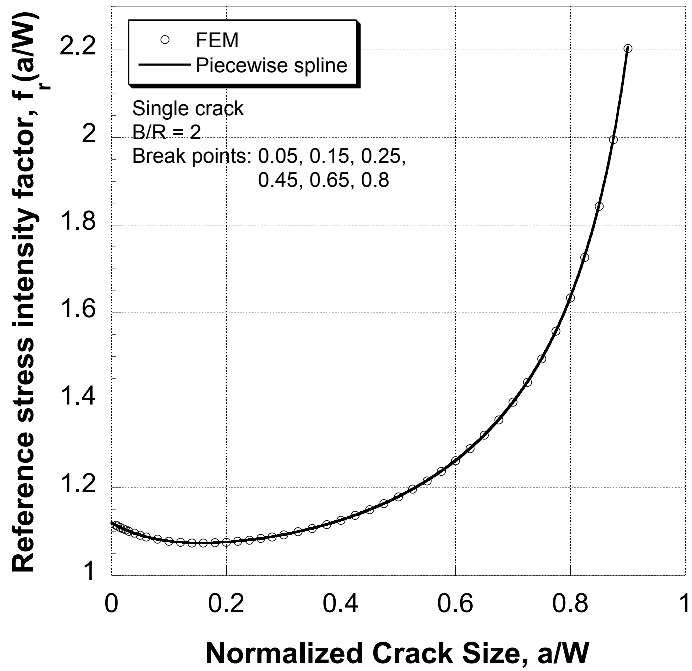
$$f_r(a/W) = \sum_{i=0}^{I} \alpha_i \left(\frac{a}{W}\right)^i$$

Single crack (Long strip with a hole, H/B ≥ 2)								
B/R	Polynomial	Interval	α_0	α_1	α_2	α_3	α_4	α_5
2	1	0 < a/W < 0.05	1.1200125	-0.80763166	7.9355633	-100.81862	994.09834	-4055.5336
	2	0.05 < a/W < 0.15	1.1187217	-0.67855268	2.7724042	2.4445643	-38.533495	74.993714
	3	0.15 < a/W < 0.25	1.1269285	-0.95211242	6.4198673	-21.871857	42.521242	-33.079269
	4	0.25 < a/W < 0.45	1.0933887	-0.28131771	1.0535096	-0.40642595	-0.40961970	1.2654203
	5	0.45 < a/W < 0.65	0.95083523	1.3026101	-5.9861697	15.237306	-17.791544	8.9907200
	6	0.65 < a/W < 0.80	-16.954556	139.03639	-429.78240	667.23151	-519.32554	163.30887
	7	0.80 < a/W < 0.90	-1164.0292	7308.2530	-18352.824	23071.033	-14521.702	3663.9029

- Tabulated beta functions

$\beta_i(a/W)$ for a single crack in a long strip

a/W	$\beta_1(a/W)$	$\beta_2(a/W)$	$\beta_3(a/W)$
		B/R = 2	
0.01	2.00000	1.28379	0.21892
0.10	2.00000	1.07432	0.18230
0.20	2.00000	1.06110	0.18035
0.30	2.00000	1.18123	0.16694
0.40	2.00000	1.44332	0.09990
0.50	2.00000	1.89209	-0.05531
0.60	2.00000	2.65011	-0.39985
0.70	2.00000	3.98797	-1.14980
0.80	2.00000	6.85984	-3.28770
0.90	2.00000	15.8901	-12.0033



Validation of the new weight function

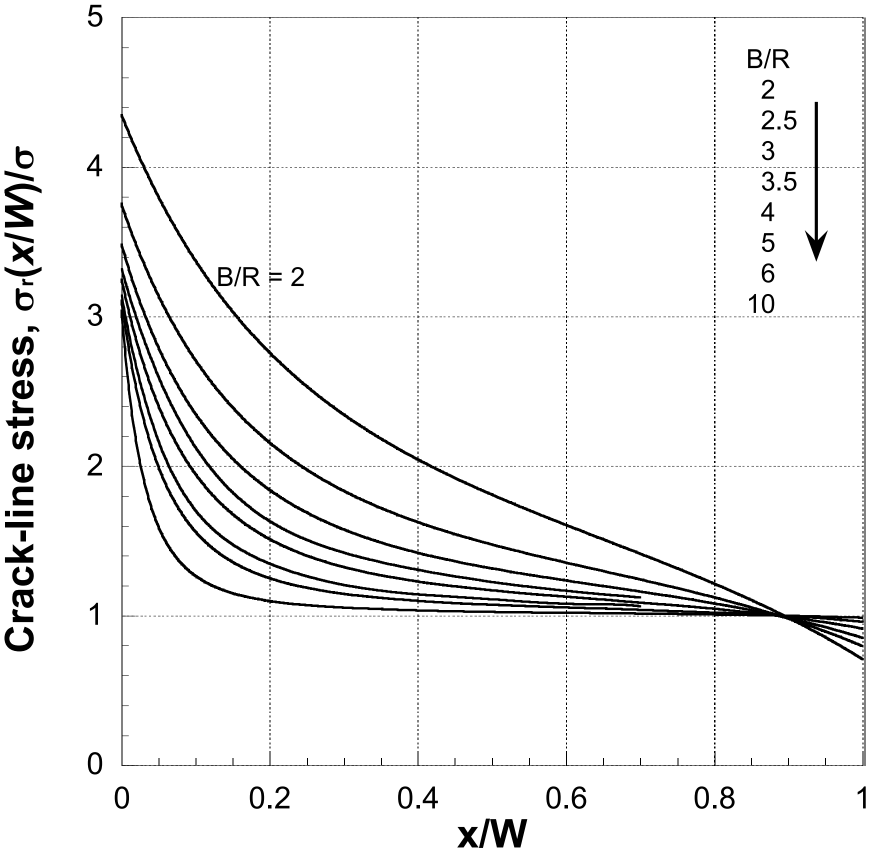
- **Validate for two cases**
  - Uniform pressure (NASSIF)
  - Remote load (NASSIF, AFGROW)

$$K^{(2)}(a) = \int_0^a \sigma^{(2)}(x) \cdot m(a,x)\, dx$$

$$m(a,x) = \frac{1}{\sqrt{2\pi a}} \sum_{i=1}^{3} \beta_i(a/W) \cdot \left(1 - \frac{x}{a}\right)^{i-\frac{3}{2}}$$

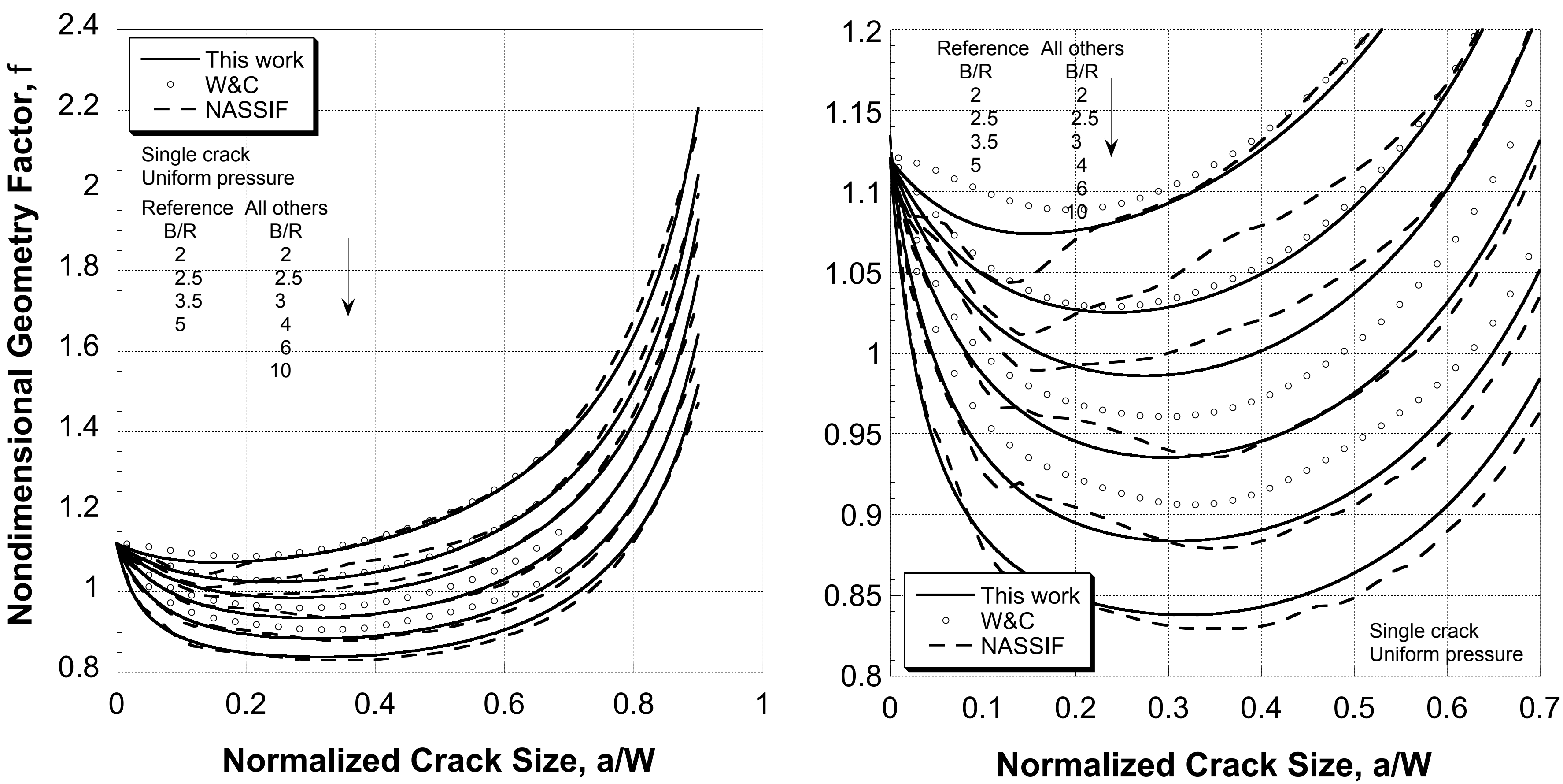
  - For integration, use Gauss-Kronrod nested quadrature (handles singularity)
  - For beta functions, use provided table a/W = 0, 0.001, …, 0.9

% beta_2(a/W) for a single crack in a long strip wi[t]			
% a/W	B/R = 2	B/R = 2.5	B/R = 3
0	1.33206	1.33131	1.332
0.001	1.32642	1.32333	1.321
0.002	1.32078	1.31535	1.310
0.003	1.31569	1.30791	1.300
0.004	1.31081	1.30066	1.290
0.005	1.30607	1.29354	1.281



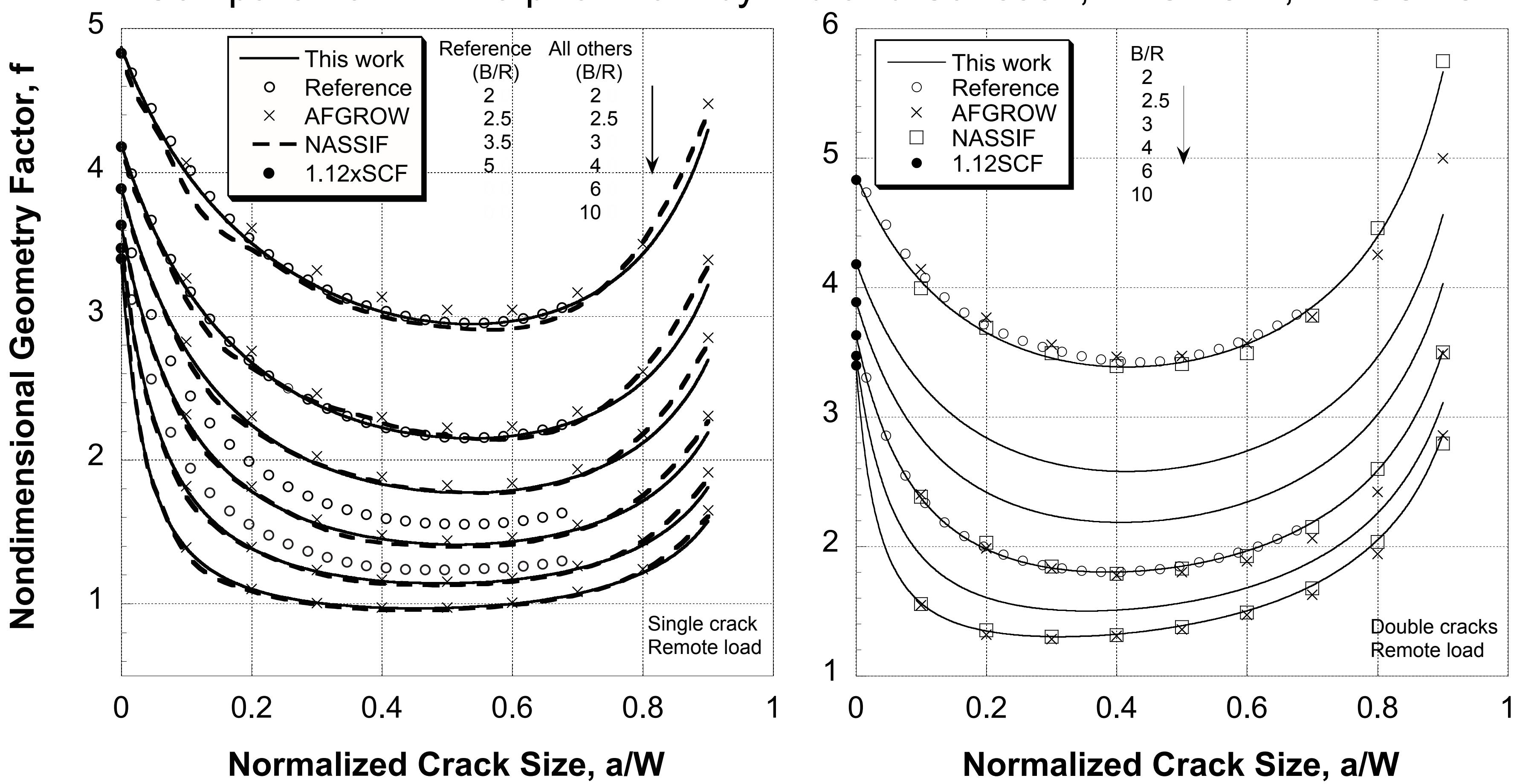
Validation of the new weight function

- **Uniform pressure: all a/W (left) and small a/W (right)**
  - Compare new work to prior work by Wu and Carlsson



Validation of the new weight function

- **Remote load: single crack (left) and double cracks (right)**
  - Compare new WF to prior work by Wu and Carlsson, AFGROW, NASGRO



Summary

- Developed a new weight function for crack at a central hole
  - Single crack
  - Double crack
  - Long strip
  - Square plate
  - Range of hole size: B/R = 2, 2,5, 3, 4, 5, 6, 10
- Validated the new weight function against handbook solutions
- Published results, with tabulation to enable direct use

Weight functions for a finite width plate with a radial crack at a circular hole

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